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Algebra I

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Algebra I Workbook

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Note: Additional paper and access to the DVDs are necessary for working these problems.

QUESTIONS

Lesson 1
An Overview

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. The statement of the think-of-a-number problem we saw at the end of the lesson went as follows:

Think of a number.
Add 4.
Multiply by 3.
Subtract 9.
Multiply by 2.
Divide by 6.
Subtract the original number.

a) What did you obtain?

b) Can you show that no matter what the starting number x, the answer will always be 1?

2. Make up a similar problem that will give 5 no matter what the starting value.

3. Select a 3×3 array of numbers from a calendar (any month, any year).

a) Add up the nine numbers.

b) Divide the sum by the central number. What do you obtain?

c) Can you show that no matter what the 3x3 array of numbers, the answer will always be 9?

II. SUPPLEMENTARY EXERCISES

1. Think of a number.
Add 1.
Multiply by 9.
Add the original number.
Subtract 4.
Delete the ones digit.

a) What do you obtain?

b) Do you think that you will always obtain the original number?

c) Can you prove your answer?

2. The following are some common errors students make in algebra. Correct the following errors by writing a true equation in each case.

a) $\frac{1}{x}+\frac{1}{y}=\frac{2}{x+y}$

b) $\frac{x+y}{w+z}=\frac{x}{w}+\frac{y}{z}$

c) $(a+b)^2=a^2+b^2$

d) $x^2 \bullet x^3=x^6$

e) $a^{-1}=-a^1$

f) $(-x)+(-x)=+2x$

III. INVESTIGATIVE PROBLEM

1. The paradox illustrates how fallacies can arise in algebra when algebraic operations are applied incorrectly. Find the source of the error.

Let $x=y$	:	$x=y$
Multiply both sides by x	:	$x^2=xy$
Subtract y^2 from both sides	:	$x^2-y^2=xy-y^2$
Factor*	:	$(x+y)(x-y)=y(x-y)$
Divide both sides by $(x-y)$	:	$x+y=y$
Since $x=y$, substitute y for x	:	$2y=y$
Divide both sides by y	:	$2=1$

* You will see in Lesson 3 that $x^2-y^2=(x+y)(x-y)$

Lesson 2
The Evolution of Numbers

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. Briefly explain what is meant by the following words or phrases:

   a) Enumeration

   b) Numeration

   c) Number

   d) Numeration Systems

   e) Number Systems

2. Give the equivalent of the following Roman numerals in our present-day base-10 system.

   a) *X* f) *I*

   b) *D* g) *V*

   c) *C* h) *CLV*

   d) *L* i) *MDIII*

   e) *M* j) *XVIII*

3.
   a) Define the set of natural numbers **N**.

   b) The natural numbers, together with their negatives and 0, comprise what number set?

   c) Define the set of rational numbers **Q**.

   d) The rational numbers, together with the irrational numbers, comprise what number set?

4. We saw that the solution of an equation depends on the nature of the variable in the equation. Answer the following questions with *Yes* or *No* and explain your answer.

a) Does $x - 3 = 1$ have a solution in **N**?

b) Does $x + 3 = 1$ have a solution in **N**?

c) Does $2x = 1$ have a solution in **Z**?

d) Does $3x = 1$ have a solution in **Q**?

e) Does $x^2 = 3$ have a solution in **Q**?

f) Does $x^2 = 5$ have a solution in **R**?

II. SUPPLEMENTARY EXERCISES

1. Express the following rational numbers as repeating or terminating decimals.

a) 2/3 d) 10/7

b) –5/8 e) –75/8

c) 2/11 f) 10/27

2. Express the following irrational numbers as non-repeating and non-terminating decimals (include 9 digits beyond the decimal point).

a) π

b) e

c) $\sqrt{2}$

d) $-\sqrt{7}$

e) $3\sqrt{3}$

3.

a) Are addition and multiplication <u>commutative</u> in **R**? (i.e.: Is $x + y = \ y = x$ and $xy = yx$ true for all real numbers x and y?

a) Are subtraction and division <u>commutative</u>? Explain your answer.

4.

a) Are addition and multiplication associative in **R**? (i.e.: Is $x + (y + z) = (x + y) + z$ and $x(yz) = (xy)z$ true for all real numbers x, y, and z?)

b) Are subtraction and division associative? Explain your answer.

5.

a) Is multiplication distributive over addition in **R**? (i.e.: Is $x(y + z) = xy + xz$ true for all real numbers x, y, and z?)

b) Use the distributive law to compute the following. Verify your answer using another method.

i) $3(4 - 3)$ ii) $-5(5 + 10)$ iii) $13(12 - 8)$

III. INVESTIGATIVE PROBLEM

1. Investigate **identity** elements and **inverses**.

a) Is there a real number a that has the property $x + a = a + x = x$ for all real numbers x? If so, this special number is called the additive identity in **R**.

b) Is there a real number m that has the property $xm = mx = x$ for all real numbers x? If so, this special number is called the multiplicative identity in **R**.

c) By now you know that the additive identity in **R** is zero. Does every real number x have an additive inverse, x' (x prime), such that $x + x' = x' + x = 0$? If so, define x' in terms of x.

d) By now you know that the multiplicative identity in **R** is 1. Does every real number x have a multiplicative inverse x'' such that $x x'' = x'' x = 1$? If so, define x'' in terms of x.

Lesson 3
The Language of Algebra

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. We saw that the general form of a monomial is ax^n, where a is a real number and n is a whole number. a is called the numerical coefficient and n is called the degree of the monomial. Specify the values of a and n in the following monomials.

a) -3

b) x

c) $2x$

d) $\frac{x}{3}$

e) $\frac{-4x^2}{5}$

f) x^3

g) $-1.5x^4$

h) $-x^5$

2. In each of the following cases, add the given monomials and specify if the sum is another monomial or a polynomial. Explain your answer.

a) $4x, -x, -5x$

b) $x^2/2, 3.5x^2, -x^2$

c) $4x, -2x^2, 6$

d) $x^2, 2xy, y^3$

3. In each of the following cases, multiply the given monomials.

a) $(1/2)x, 2x$

b) $3x^2, 2x^3$

c) $-7y, -4y^4$

d) $\frac{z}{3}, \frac{-2z^2}{3}, 9z^3$

4. In each of the following cases, divide the first monomial by the second. Specify if the quotient is a monomial or not. Explain your answer.

a) $4x^2, 2x$

b) $2x^3, 4x^3$

c) $\frac{x^3}{2}, \frac{x^2}{3}$

d) $-x, -5x^3$

II. SUPPLEMENTARY EXERCISES

1. Perform the indicated operations.

a) $(x^2 + 2x + 1) + (-2x^2 - 3x)$

b) $(x^2 + 2x - 1) - (4x^2 - 7x + 2)$

c) $8x(x^2 - 4)$

d) $(2x + 1)(3x + 7)$

e) $(3x + 1)^2$

f) $(4 - x)(4 + x)$

2. In Column A, the expressions are in the <u>factored form</u>; in Column B, they are in the <u>"multiplied out" form</u>, or simply, the <u>polynomial form</u>. Pair up the expressions that are equivalent (example: a and 3).

<u>Column A</u>	<u>Column B</u>
a. $(x + 3)(x - 4)$	1. $12x^2 - 6x - 6$
b. $(x - 3)(x + 4)$	2. $x^2 + x - 12$
c. $(x - 2)^2$	3. $x^2 - x - 12$
d. $(x + 4)(x - 4)$	4. $4x^2 + 12x + 9$
e. $(3x - 3)(4x + 2)$	5. $x^2 - 4x + 4$
f. $(2x - 1)(2x + 1)$	6. $4x^2 + 16x + 16$
g. $(2x + 3)^2$	7. $x^2 - 16$
h. $(1 - x)(2 + x)$	8. $-x^2 - x + 2$
i. $(4 + 2x)^2$	9. $4x^2 - 1$

3. Factor the following polynomials.

a) $4x - 2x^2$

b) $4xy^2 + 6x^2y$

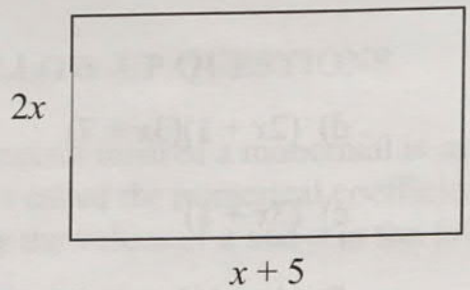
c) $16x^2 - 4$

d) $4x^2 + 20x + 25$

e) $4x^2 - 4x + 1$

III. INVESTIGATIVE PROBLEM

1. The length and width of a rectangle are as shown:



a) Write a polynomial that expresses the perimeter of the rectangle.

b) Write a polynomial that expresses the area of the rectangle.

c) Write a polynomial that expresses one-half of the perimeter.

d) Write an equation that states that the perimeter is 70.

e) Write an equation that states that the area is 300.

Lesson 4
Exploring Functions with the Aid of Graphing Calculators

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. Among the following, pick out the equations that are to be solved for the unknown variable. Identify the remaining equations as a formula, an identity, or a property of real numbers.

a) $x+y=y+x$

b) $y+1=10$

c) $P=2l+2w$

d) $(x+y)(x-y)=x^2-y^2$

e) $2x=1$

f) $A=\pi r^2$

2. We saw the formula for the area of a triangle $A=(1/2)bh$. Evaluate A for $b=2.5$ and $h=4.2$.

3. We saw the identity $(x+y)(x-y)=x^2-y^2$. Rewrite this identity with $x=4$ and $y=2t$.

4. We saw the property $x(1/x)=1, x\neq 0$. Give a few instances of this property.

5. We saw the function $y=5.50x$ where x was the number of babysitting hours and y, the corresponding earnings in dollars.

a) According to this equation, what would you earn for $x=6.5$ hours of babysitting?

b) How many hours would you have to babysit to earn $55.00?

II. SUPPLEMENTARY EXERCISES

1. We saw that the equation for the x-axis (the horizontal/input axis) is $y=0$.

a) Can you make sense of this equation?

b) Can you deduce the equation of the horizontal line passing through (0,3)? Explain your equation.

2. We saw that the equation for the y-axis (the vertical/output axis) is $x=0$.

a) Can you make sense of this equation?

b) Can you deduce the equation of the vertical line passing through (–2,0)? Explain your equation.

3. Explore the equation $y = x^2$ (with the aid of your graphing calculator).

a) Press "y=" and key in "x" "x^2" to obtain "$y = x^2$".

b) Press WINDOW and key in $x_{min} = -5$, $x_{max} = 5$, $x_{scl} = 1$, $y_{min} = -2$, $y_{max} = 25$, and $y_{scl} = 1$.

c) Press GRAPH. Describe the shape of this graph. Explain the meaning of this infinite set of points in the xy-plane.

d) Press TRACE and with the aid of the right and left arrow keys, trace the graph of $y = x^2$. Find the approximate y-value that corresponds to $x = -3$ (you may ZOOM IN). Find the approximate x-values that correspond to $y = 20$ (you may ZOOM IN).

e) Press "2nd" WINDOW to obtain TABLE SETUP. Set TblMin = –5 and Δ Tbl = .1; then press "2nd" GRAPH to obtain TABLE. Find the exact y-value that corresponds to $x = -2.5$. Find the exact x-values that correspond to $y = 2.25$.

III. INVESTIGATIVE PROBLEM

1. Practice storing and plotting data.

a) Press STAT, select 1. Edit and enter the following data in L_1 and L_2.

Shoe Length (cm) x	Arm Length (cm) y
25	36
31	50
24	32
27	40
21	28
22	30
28	46
29	46
32	49

b) Press "2nd" "y=" to obtain STAT PLOTS. Select 1: Plot 1. Turn it on. For: Type: Select the first icon (Scatter plot).

Xlist: Select L1 (since the x-values are stored in L1).
Ylist: Select L2 (since the y-values are stored in L2).
Mark: Select the mark of your choice.

c) Press ZOOM and select 9: ZoomStat. (Make certain that your entries in the "y=" menu are either deselected or cleared.)

d) What do you observe about the nine (x,y) data points? Press TRACE and the right arrow key to verify that all nine pairs of numbers have been correctly keyed in.

e) We will explore these types of problems in more depth in Lesson 10.

Lesson 5
Linear Functions—Introductory Explorations

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. Look up René Descartes in an encyclopedia and pick out three or four important things for which he is remembered.

2. Using the formula for the *n*th positive even number, find the 43^{rd} even number.

3. Using the same formula as in #2, determine the ordinal number associated with the even number 176.

4. Using the formula for the *n*th odd number, find the 26^{th} odd number.

5. Using the same formula as in #4, determine the ordinal number associated with the odd number 99.

6. Using the linear function for Reed's Reasonable Rates (RRR), determine the amount of money (in dollars) you would have to pay Reed for 3.75 hours of work.

7. Using the same function as in #6, determine the number of hours Reed worked for a pay of $170.00.

8. Using the linear function for the Interest-Free Loan (IFL), determine your balance after 10 payments.

9. Using the same function as in #8, determine the number of payments you have already made if your balance is now $124.00.

10. Explain in your own words why the graph for RRR "rises" from left to right, whereas the graph of IFL "falls."

11. Can you predict, from the four linear functions you saw in this lesson, the general form for the equation of a linear function? (This will be explained in Lesson 7.)

II. SUPPLEMENTARY EXERCISES

1.

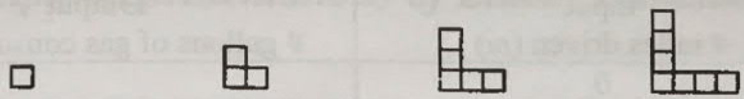


Figure #(n) **1** **2** **3** **4** ••• ***n*th**

Area (A):

a) Find the area of the first four figures of the sequence illustrated above.

b) Can you generalize the area for the nth figure?

c) Do you think the area A is a <u>linear</u> function of the figure number n? Why?

2. Answer the same three questions as in problem II.1 (above) for the two sequences that follow.

a)

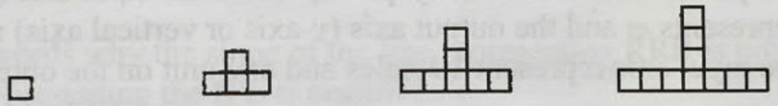


Figure #(n) **1** **2** **3** **4**

Area (A):

b)

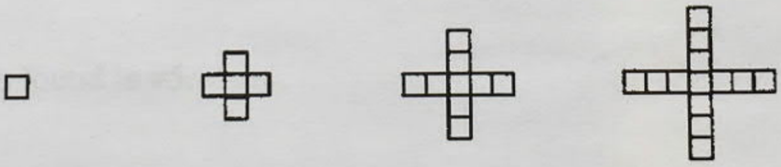


Figure #(n) **1** **2** **3** **4**

Area (A):

3. Consider the above three patterns together. What observations can you make?

III. INVESTIGATIVE PROBLEM

1. Sandra's car gets 22 miles per gallon of gas.

a) Complete the following input/output table.

Input # miles driven (m)	Output # gallons of gas consumed (g)
0	
22	
44	
66	
88	
• • •	
m	

b) Use this table of numerical values to derive the linear function relating g, the number of gallons consumed, and m, the number of miles driven.

c) Graph the function in the xy-plane, where the input axis (x-axis or horizontal axis) represents m and the output axis (y-axis or vertical axis) represents g. (Let one unit on the input axis represent 10 miles and one unit on the output axis represent 1 gallon.)

d) Use the graph to estimate the number of gallons of gas necessary for a 250-mile trip.

e) Use the graph or the equation to determine the number of miles traveled if 15 gallons of gas were consumed.

Lesson 6
Multiple Representations of Linear Functions

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. Compute the slope of the graph for Reed's Reasonable Rates (RRR) using:

   a) The points (0,45) and (4,145)
   b) The points (1,70) and (3,120)
   c) What do you notice?

2. Compute the slope of the graph for the Interest-Free Loan (IFL) using:

   a) The points (3,196) and (14,–2)
   b) The points (0,250) and (2,214)
   c) What do you notice?

3. Explain in your own words why the slope of the line representing RRR is positive, while the slope of the line representing the IFL is negative.

4. Make an input/output table for RRR in which the input variable varies from 0 to 5 hours in half-hour increments.

5. Suppose Reed's hourly rate increased to \$35.00/hr and the cost of his home visit to \$50.00. Write the new equation for the cost (y) in dollars as a linear function of the time (x) in hours.

6. Graph the function you found in #5.

   a) What is the slope?
   b) What is the y-intercept?
   c) What does this point represent?

7. Suppose you borrowed \$300.00 and had to reimburse this interest-free loan in monthly payments of \$25.00. Write the new equation for the balance (y) in dollars as a linear function of the number of payments (x).

8. Graph the function you found in #7.

   a) What is the slope?
   b) What is the y-intercept?
   c) What does this point represent?

d) Can you explain the meaning of the point (12,0)?

II. SUPPLEMENTARY EXERCISES

1. A line l passes through the points (0,3) and (10,15).

a) What is the slope of the line l?
b) What is its y-intercept?
c) What is the algebraic equation of the linear function whose graph is the line l?
d) Generate an input/output table for x-values (input values) between –4 and +4.

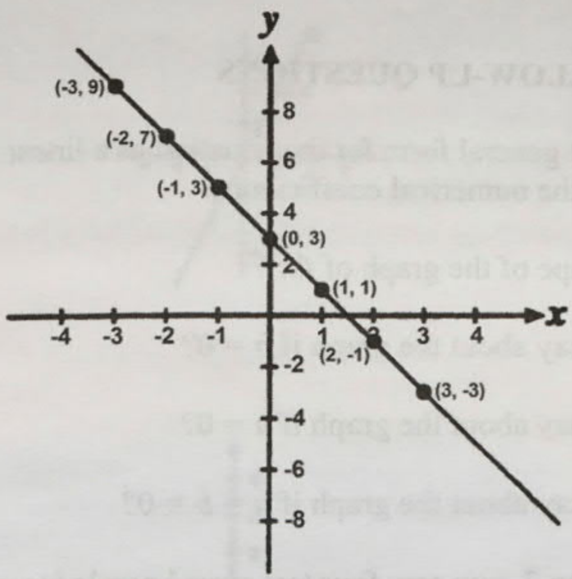
3.Let $f(x)$ be the linear function having the following input-output table.

Input (x)	Output (y)
1	2.5
2	5
3	7.5
4	10
5	12.5
6	15

a) What is the slope?
b) What is its y-intercept?
c) What is the algebraic equation of $f(x)$?
d) Can you think of a situation that could fit these numbers?
e) Graph $f(x)$.

4.

a) Derive the algebraic equation of the linear function whose graph is the following:



b) Describe the "staircase effect" in this case.

III. INVESTIGATIVE PROBLEM

1. The algebraic equation $w = 5.5h - 220$ expresses the relationship between the "normal" weight (w) of an adult male in pounds and his height (h) in inches.

   a) Letting h be the input-axis and w the output-axis, graph this function.
   b) Generate an input/output table for this function in which the input variable h varies from 60 to 78 inches in 2-inch increments.
   c) Use either representation to determine the "normal" weight of a six-foot-tall adult male.
   d) Use either representation to determine the height of an adult male whose normal weight is 198 pounds.
   e) Explain the meaning of the slope (5.5).

Lesson 7
The Geometry of Linear Function Graphs

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. You now know the general form for the equation of a linear function: $f(x) = y = ax + b$, where a and b are the numerical coefficients.

   a) What is the shape of the graph of $f(x)$?

   b) What can you say about the graph if $b = 0$?

   c) What can you say about the graph if $a = 0$?

   d) What can you say about the graph if $a = b = 0$?

2. At the end of Lesson 7, you saw four test-your-knowledge exercises on a transparency. They are transcribed below for your convenience. In each case, make two statements about the graph(s):

   a) $y = 3x - 5$

   b) $y = -3x$

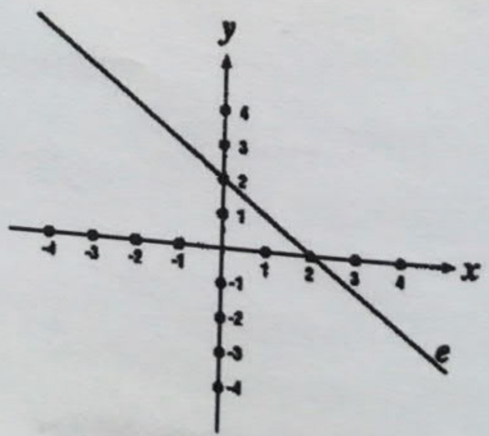
   c) $y = 6x + 3$
   $y = 6x - 1/3$

   d) $y = 7x$
   $y = (-1/7)x + 1$

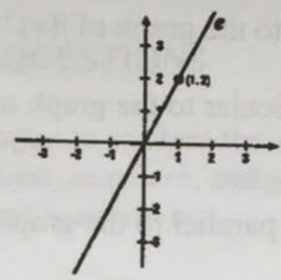
II. SUPPLEMENTARY EXERCISES

1. In each of the following cases, make two statements about the numerical coefficients a and b of the algebraic equation(s) ($f(x) = y = ax + b$) corresponding to the line(s):

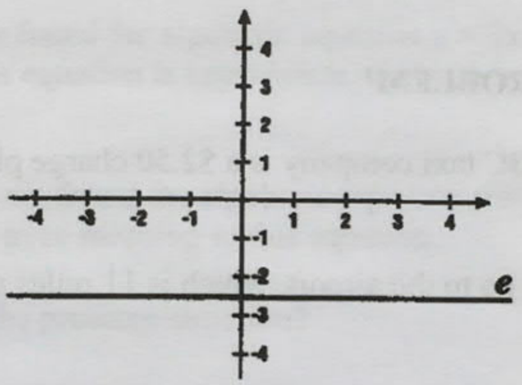
   a)



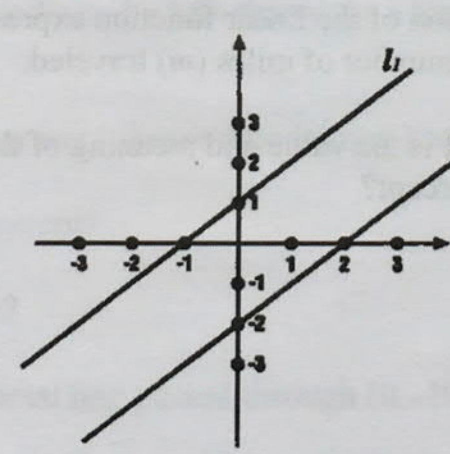
b)



c)



d)



2. Make three statements about the graphs of:

$f(x) = y = 0x + 7$ (or $f(x) = y = 7$), and

$g(x) = y = 0x - 3.5$ (or $g(x) = y = -3.5$)

3. Let $f(x) = y = -0.2x + 5$ be a linear function.

   a) How many lines are parallel to the graph of $f(x)$? What do they have in common?

   b) How many lines are perpendicular to the graph of $f(x)$? What do they have in common?

   c) Give the equation of the line parallel to the graph of $f(x)$ and passing through (0,–4).

   d) Give the equation of the line perpendicular to the graph of $f(x)$ and passing through (0,5).

III. INVESTIGATIVE PROBLEM

1. The night rate of the ABC taxi company is a \$2.50 charge plus \$0.60 for each mile traveled.

   a) Suppose you had to go to the airport, which is 11 miles away. What would an ABC taxi ride cost you?

   b) Your friend paid an ABC taxi \$5.50 last night. How long was her trip (in miles)?

   c) Write the algebraic equation of the linear function expressing the night fare (F) of an ABC taxi in terms of the number of miles (m) traveled.

   d) Graph this function. What is the value and meaning of the slope? What is the value and meaning of the y-intercept?

Lesson 8
Words, Equations, Numbers, and Graphs

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. In Lesson 8, you were encouraged to explore the various forms of representation of linear functions called *words*, *equations*, *numbers*, and *graphs*. Let $f(x)$ be a linear function; explain in your own words what is meant by:

a) *Words*

b) *Equation*

c) *Numbers*

d) *Graph* of the function $f(x)$

2. In the first activity, we found the algebraic equation $y = 7x$. We invented one problem situation for which this equation is appropriate. Can you invent another? (**Note**: Clearly define the variables.)

3. In the second activity, we found the algebraic equation $y = 1.5x + 3$. We invented a problem situation that gave meaning to this equation.

a) Can you reiterate the problem situation?

b) Suppose you made a 20-minute long distance call, what would it cost?

c) Suppose your bill indicated a \$48 charge for a long distance call, how long was it (in minutes)?

4. In the final activity, you were shown a horizontal line whose equation was $y = 20$.

a) What was its y-intercept?

b) What was its slope?

c) Suppose the horizontal line passed through $(0,-10)$ instead. Derive its algebraic equation.

d) Suppose the horizontal line passed through the origin $(0,0)$. Derive its algebraic equation.

e) Conclude with the general algebraic equation of a linear function whose graph is a horizontal line.

II. SUPPLEMENTARY EXERCISES

1.

a) Derive an algebraic equation from the following table of numbers and invent a hypothetical situation that gives meaning to the equation you found.

x	y
1	4
2	6
3	8
4	10
5	12

b) What is the y-intercept? Does it make sense?

2. Graph → Input/Output Table → Equation

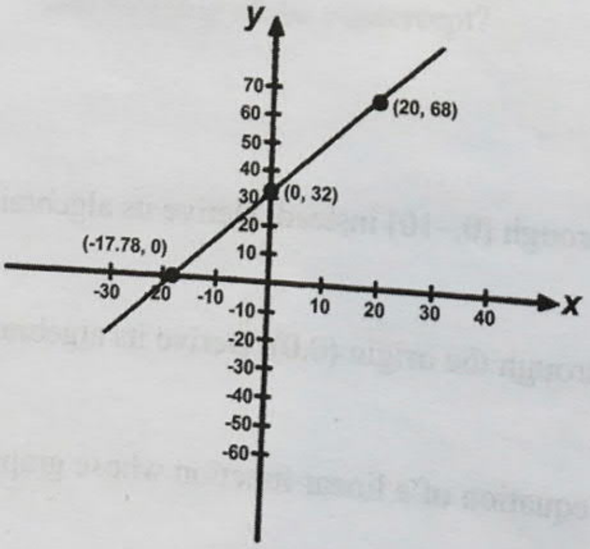
a) Suppose I showed you a linear graph (i.e., a line) on my calculator screen. How would you obtain an input/output table of numbers from that graph?

b) How would you obtain the algebraic equation from this table of numbers?

III. INVESTIGATIVE PROBLEM

1. Graph → Equation → Problem

a) A linear function has the following graph:



b) We now know that $f(x) = ax + b$ is the general form of a linear function. Can you derive the value of b from the graph above?

c) Can you derive the value of a from the graph above?

d) Using your answers to questions b) and c), write the algebraic equation of this linear function.

e) This is a well-known relationship. Do you recognize it? If so, explain it. (**Note**: Define the variables clearly.)

Lesson 9
Problem Solving with Linear Equations

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. In problem 1, we found $y = 4x + 4$.

   a) What did the variables x and y represent?

   b) How many tiles are in the frame of figure #19?

   c) Which figure has a frame of 160 tiles?

2. In problem 2, we found $y = -240x + 15{,}000$.

   a) What did the variables x and y represent?

   b) How many hours would it take this pump to empty the pool halfway?

3. What is meant by the *zero* or the *root* of a linear function $f(x) = ax + b$? Can the zero equal zero? Give an example.

4. How many zeros (or roots) does a linear function of the form $f(x) = ax + b$ have?

5. Where, on its graph, do we find the zero of a linear function?

II. SUPPLEMENTARY EXERCISES

1.

Figure #:	1	2	3	4
Partial Frame:	5	8	11	14

In this sequence of figures, the "partial frame" consists of the tiles bordering the darkened square on <u>three</u> sides.

a) Find the linear relationship between the figure number x and the number of tiles y in the partial frame.

b) A figure in this sequence has a partial frame of 83. What is the figure number?

2. A large bucket contains 5 liters of water. More water is then added at a rate of 4.5 liters per minute.

a) Write the linear relationship between the number of minutes x and the corresponding volume y of the bucket, in liters.

b) How long (in minutes) will it take to fill the bucket (maximum capacity 95 liters)?

3. Solve the following equations for x.

a) $3x - 4 = x + 7$

b) $-0.5x + 7 + 2x = 10$

c) $2(x - 3) = 5x - 8$

4. Find the zero (or root) of $f(x) = -5x + 7$.

III. INVESTIGATIVE PROBLEM

1. Suppose you wish to purchase a \$250 VCR, so you get a full-time summer job in a restaurant for one month. Your salary is \$5.25 per hour, but you must pay for the gas required to commute to work by car (\$20/month).

a) Derive the equation of the linear function relating the number of hours x to the money earned y (in dollars).

b) How many hours must you work in order to afford purchasing the VCR?

c) Explain how you solved this problem by functional exploration.

d) Show how you solved this problem by symbol manipulation.

Lesson 10
Modeling Real-World Data with Linear Functions

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. In the first modeling problem, we collected the following data in our mathematics laboratory:

Number of Drops x	Avg. Diameter of Ink Blot (in cm) y
1	2.1
2	2.8
3	3.5
4	4.1
5	4.6

a) An *eyeball-fit line* is a line that "seems to fit" the data. We spoke of it but did not take the time to compute it. Here is your opportunity. Plot these points and draw the line that passes through (1,2.1) and (4,4.1), as I illustrated with the uncooked spaghetti string. Find the equation of this line. How does it compare with the linear-regression line (or best-fit line) we found ($y = .63x + 1.53$)?

b) I also spoke of another good-fit line involving the computation of several slopes and their average. Begin by computing the slope of the line joining:

(1,2.1) and (2,2.8)
(2,2.8) and (3,3.5)
(3,3.5) and (4,4.1)
(4,4.1) and (5,4.6)

Then compute the average of these four slopes and call it a. Finally, use any of the five points to find b. Compare this good-fit line ($y = ax + b$) with the best-fit line $y = .63x + 1.53$.

2. In the second modeling problem, we used the following previously collected data:

Mileage x	Cost ($) y
2,000	16,300
4,500	16,200
5,000	15,700
10,500	14,500
12,000	14,100
14,000	13,700
16,500	12,600

a) Plot these points and draw an eyeball-fit line. Find its equation.

b) Select a few pairs of points. Compute their corresponding slopes. Derive a good-fit line.

c) Compare the equations you obtained in a) and b) with the best-fit line we found ($y = -.25x + 17{,}057$).

II. SUPPLEMENTARY EXERCISES

1. Describe in your own words the meaning of the slope .63 in our first mathematical model $y = .63x + 1.53$.

2. Same question for the slope –.25 in our second mathematical model $y = -.25x + 17{,}057$.

3. Select four or five objects that have varying heights. Take them outside on a sunny day. Measure the length of each object's shadow at the same time of day. Organize your collected data into the following table (both x and y are measured in inches).

	Object's Height (x)	Object's Shadow Length (y)
Object 1		
Object 2		
Object 3		
Object 4		
Object 5		

III. INVESTIGATIVE PROBLEM

1. The following questions will be based on the data you collected in exercise II.3.

   a) Show that the length of the shadow is a <u>linear</u> function of the height of the object casting the shadow.

   b) Key the data you collected into your graphing calculator's lists. For example, the x-values in L_1 and the y-values in L_2.

   c) Use linear regression to find the best-fit line.

   d) Use the mathematical model you found in III.1.c) to estimate the shadow cast by a 15-foot tree (at the same time of day).

Lesson 11
Linear Functions and Geometry

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. In the first exploration, we measured the diameter and circumference of five circles. We obtained the following data:

Circle #	D	C
1	4.1	13
2	7.5	23.4
3	9.25	29.4
4	11.75	37.2
5	15.2	47.5

a) Let $D = x$ and $C = y$. Compute the ratio (change in y)/(change in x) for circles 1 and 2, 2 and 3, 3 and 4, and 4 and 5.

b) Compute the average of these four ratios. How accurate is this slope?

c) Recall the linear relationship between D and C (or x and y). Explain why it is linear.

2.

a) Define the meaning of π in your own words.

b) How many times does the diameter of a circle fit around its circumference?

c) How many times does the radius of a circle fit around its circumference?

3. In the second exploration, we discovered a linear relationship between the sum of the interior angles of a (convex) polygon, A, and the number of sides of the polygon, n.

a) Can you recall this linear relationship?

b) Show that it is, indeed, linear.

c) What is the sum of the interior angles of an octagon?

d) Suppose the sum of the interior angles of a polygon is 2340°. How many sides does this polygon have?

II. SUPPLEMENTARY EXERCISES

1. Let S denote the side length of a square.

   a) Express its perimeter P as a function of S.

   b) Is this a linear relationship? Explain.

   c) Express its area A as a function of S.

   d) Is this a linear relationship? Explain.

2. Find other linear relationships in geometry.

3. A rectangle is 20 feet longer than it is wide. Let w denote the width, l the length, and P the perimeter of the rectangle.

   a) Write an algebraic expression for the length of the rectangle in terms of w. Is this a linear relationship? Explain.

   b) Write an algebraic expression for the perimeter of the rectangle in terms of w. Is this a linear relationship? Explain.

   c) What is the length if the width is 5?

   d) What is the perimeter if the width is 10?

   e) What are the width and length if the perimeter is 44?

III. INVESTIGATIVE PROBLEM

1. Three tennis balls are packaged snugly inside a cylindrical box such that there is no room for them to move around.

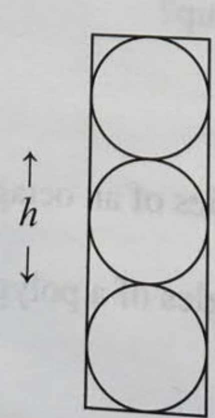


Figure: Section of cylindrical box and balls.

a) Without calculating, which do you predict is longer—the height h of the box or its circumference c?

b) Let d be the common diameter of the three spherical balls. What is the circumference c of the box in terms of d?

c) What is the height h of the box in terms of d?

d) What is your conclusion?

Lesson 12
Quadratic Functions—Introductory Explorations I

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. What is the 25th square number?

2. What is the 15th oblong number?

3. What is the 10th "star-like" number?

4. What is the quadratic term in the algebraic equation of:

   a) the *n*th square number?

   b) the *n*th oblong number?

   c) the *n*th "star-like" number?

5. What is the linear term in the algebraic equation of:

   a) the *n*th square number?

   b) the *n*th oblong number?

   c) the *n*th "star-like" number?

6. What is the constant term in the algebraic equation of:

   a) the *n*th square number?

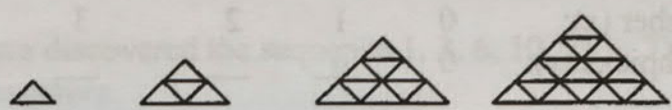
   b) the *n*th oblong number?

   c) the *n*th "star-like" number?

7. What is the standard-form equation of a quadratic function? Indicate the quadratic term, the linear term, and the constant term.

II. SUPPLEMENTARY QUESTIONS

1. Counting Triangles. Unit triangles Δ are used to form larger triangles. Consider the pattern below:



n: # of rows 1 2 3 4 •••

T: # of unit triangles 1 ___ ___ ___ •••

   a) Fill in the blanks above with the number of unit triangles in each figure.

   b) Construct an input/output table for this pattern, in which the input variable is the number of rows and the output variable is the number of unit triangles.

2.

   a) Compute the first differences, D_1, in the input/output table you constructed above (II.1). What do you notice?

   b) Compute the second differences, D_2. What do you suspect?

3. For the triangle pattern above:

   a) Can you use the result from exploration 1 (in the videotape) to deduce the algebraic expression for the number of unit triangles in an n-row triangle?

   b) How many unit triangles are in a 7-row triangle?

III. INVESTIGATIVE PROBLEM

Building Squares with Toothpicks

1. Consider the following pattern formed with equal-length toothpicks.

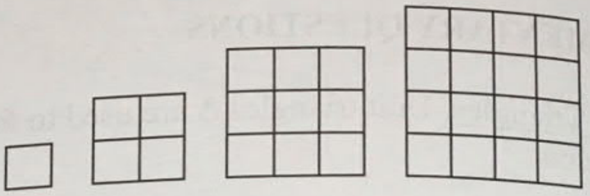


Figure number (x):	0	1	2	3	4	…
No. of toothpicks (y):	0	4	___	___	___	

a) Fill in the blanks with the number of toothpicks in each figure.

b) Make an input/output table for the relationship between the figure number (input variable) and the number of toothpicks (output variable).

c) Plot these points in an xy-plane and connect them. What can you say about the graph?

d) Compute the first differences (D_1). Can you see the connection between the first differences and the structure of consecutive figures?

e) Compute the second differences (D_2). What do you suspect?

f) <u>Challenge</u>: Can you derive the algebraic equation for the number of toothpicks in the nth figure? (If you cannot at this point, do not worry. You will learn how to do this in Lesson 14.)

Lesson 13

Quadratic Functions—Introductory Explorations II

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. In the first exploration, we discovered the sequence 1, 3, 6, 10, 15.... These numbers are often called *triangular numbers*.

   a) Why are these numbers called triangular numbers?

   b) What is the algebraic expression for the nth triangular number?

   c) Indicate the quadratic term, the linear term, and the constant term in this expression.

   d) We saw that the standard form for all quadratic functions is $f(x) = ax^2 + bx + c$. Indicate the values of a, b, and c in the expression for the nth triangular number.

   e) What is the 14th triangular number?

2. In the second exploration, we derived the quadratic function $f(x) = -x^2 + 12x$, or $y = -x^2 + 12x$, where y, the area, was a function of x, the length of the rectangular pen. We graphed this function and found a parabola.

   a) What are the values of the numerical coefficients a, b, and c in this quadratic function?

   b) Indicate the quadratic, linear, and constant terms of this algebraic expression.

   c) What was the vertex of the parabola? What does it represent?

   d) What were the x-intercepts of the parabola (the points where the parabola intersected the x-axis)? What do they represent?

II. SUPPLEMENTARY EXERCISES

1. <u>A Pattern Involving the Triangular Numbers</u>. Consider the following pattern:

$1^3 =$ ____
$1^3 + 2^3 =$ ____
$1^3 + 2^3 + 3^3 =$ ____
$1^3 + 2^3 + 3^3 + 4^3 =$ ____
$1^3 + 2^3 + 3^3 + 4^3 + 5^3 =$ ____

a) Fill in the blanks with the appropriate sums.

b) What do you notice about these sums?

c) Use the triangular numbers to express these sums.

d) Consider the generalization of this pattern. Write an English sentence to describe what we have found. Can you describe the same result with an algebraic sentence?

2. Diagonals in a Polygon

a) Begin by drawing a triangle, a quadrilateral, a pentagon, a hexagon, and a heptagon (make convex polygons, but they need not be regular).

b) Draw the diagonals in each figure.

c) Construct an input/output table in which the input is the number of sides and the output is the number of diagonals.

d) Is this a quadratic relationship? Why?

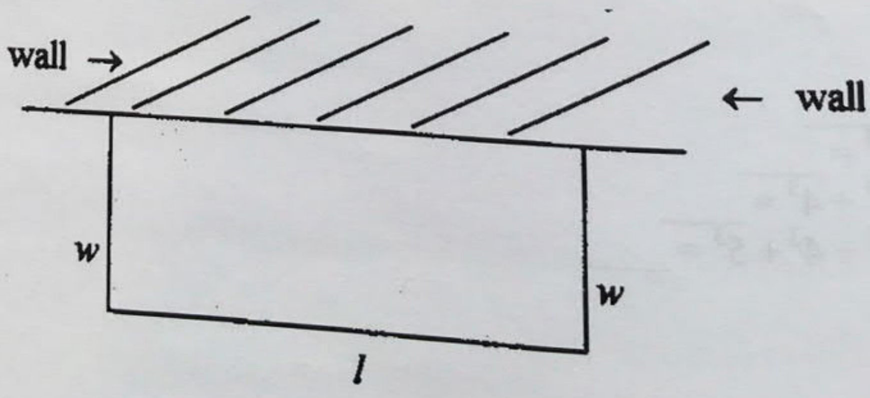
e) Challenge: Can you derive the general algebraic expression for the number of diagonals in an x-gon (an x-sided polygon)?

III. INVESTIGATIVE PROBLEM

1. A Maximization Problem. Suppose you wish to build a rectangular pen for your pet with 20 yards of fence. You will use the back wall of your house for one side; therefore, you need to use the fence on only the three remaining sides of the rectangle.

a) Draw the different possible rectangles in which both length (l) and width (w) are whole numbers. Compute the respective areas. Which configuration yields the maximum area?

b) Denote by w the two equal sides and by l the third side of the |__|–shaped fence.



Make an input/output table where the

input variable is w and the output variable is A, the area. Plot these points.

c) The area A of the rectangle is lw. Can you express A as a quadratic function of the variable w alone?

d) Graph the function you found in part c). Does the graph pass through all the points you plotted in part b)?

e) What is the vertex of the parabola? What does it represent?

f) What are the x-intercepts of the parabola? What do they represent?

Lesson 14
The Geometry of Quadratic Function Graphs

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. You now know the general form (or standard form) for the equation of a quadratic function: $f(x) = ax^2 + bx + c$, where a, b, and c are the numerical coefficients.

   a) What is the shape of the graph of $f(x)$? What is the name of such a graph?

   b) What can you say about the parabola if a is positive?

   c) What can you say about the parabola if a is negative?

   d) Explain why $(0,c)$ is the y-intercept of a parabola.

   e) What is the value of $f(x)$ (or y) at the x-intercepts? Why?

   f) What is meant by the vertex of a parabola?

2. At the end of the lesson, you saw the graphs of $y_1 = x^2$, $y_2 = x^2 - 10x + 25$, and $y_3 = x^2 + 10x + 25$ graphed sequentially.

   a) How did the graph of y_2 compare to that of y_1? Why?

   b) How did the graph of y_3 compare to that of y_1? Why?

   c) Compare the graphs of y_2 and y_3.

II. SUPPLEMENTARY EXERCISES

1. In each of the following cases, state two properties of the function's graph (check your answers by graphing).

   a) $f(x) = x^2$

   b) $f(x) = -7x^2 + 5$

   c) $f(x) = (1/25)x^2 + 2$

   d) $f(x) = 3x^2 - 3x$

2. <u>Similarities and Differences</u>. For each of the following pairs of functions, compare and contrast their graphs.

a) $f(x) = 2x^2 - 3x + 7$,
$g(x) = -2x^2 + 3x - 7$

b) $f(x) = x^2 - 4x + 4$,
$g(x) = x^2 + 4x + 4$

c) $f(x) = (1/20)x^2$,
$g(x) = 20x^2$

d) $f(x) = 3x^2 + 2$,
$f(x) = 3x^2 - 9$

III. INVESTIGATIVE PROBLEM

1. Let $f(x) = ax^2 + bx + c$. The graph of $f(x)$ passes through the points (0,0), (1,–1), (1,–1). From this information, can you deduce the values of a, b, and c and, therefore, define $f(x)$ uniquely?

Lesson 15

Words, Equations, Numbers, and Graphs

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. In the lesson, we filled out a finite-difference table for the <u>general case</u> of an arbitrary quadratic function. The table headings were:

x	$y = ax^2 + b\,x + c$	D_1	D_2

Let us take a closer look at each column to ensure a deeper understanding.

a) For the finite-difference method, what values must we have in the first column (under x)?

b) Plug in the x values, one by one (from 0 to 5), into the y expression and complete column 2.

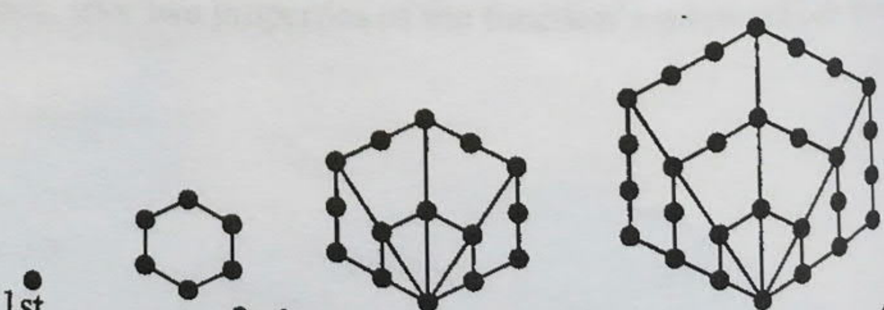
c) Fill in column 3 (D_1) by computing the differences between the consecutive y values. We call these *first differences*.

d) Fill in column 4 (D_2) by computing the differences between the consecutive first differences. We call these *second differences*.

e) Circle the three expressions that are useful to remember.

II. SUPPLEMENTARY QUESTIONS

1. Following is a pictorial representation of the first four hexagonal numbers.



Ordinal number (x): 1st 2nd 3rd 4th

Number of dots (y): 1 6 ___ ___

a) Fill in the blanks above. Can you visualize the 5th hexagonal number? If you can, give its number of dots.

b) Construct an input/output table where x is the number of dots on one side of the hexagon (it is also the ordinal number) and y is the total number of dots.

c) Compute the first differences.

d) Compute the second differences.

e) Use the results from I.1. to derive the algebraic expression for the nth hexagonal number.

f) Use the formula you found in e) to find the 10th hexagonal number.

III. INVESTIGATIVE PROBLEM

1. Suppose you have the sequence of numbers 2, 4, 12, 26, 46, 72….

a) Let 2 be the 0th number, 4 be the 1st number, 12 be the 2nd number, and so on of the sequence.

b) Make an input/output table for this sequence.

c) Compute the first differences.

d) Compute the second differences.

e) Use the finite-difference method to derive the expression for the nth number in this sequence.

f) Plot the discrete points (0,2), (1,4), (2,12)….

g) What type of a graph do you obtain?

Lesson 16
Problem Solving with Quadratic Equations

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. In problem 1 we were given $T = f(n) = (3/2)n^2 + (3/2)n$ for the number of toothpicks in figure n.

   a) Determine the figure number n such that $T = f(n) = 408$.

   b) You learned that the general form of a quadratic equation (in one unknown x) is $ax^2 + bx + c = 0$. Show that the equation in part a) fits this form. Specify a, b, and c.

   c) How many toothpicks would figure 9 require?

2. Select several common rectangular items, such as your notepad or notebook, a cereal box, a credit card, and so on, and see how close their respective ratio L/W is to the golden ratio.

3. Which specific equation is known as the *Golden Quadratic Equation*? Why?

4.

   a) What is meant by the root(s) (or zero[s]) of a quadratic function?

   b) How many root(s) does a quadratic function have?

5. Find the roots of $f(x) = -2x^2 - 7x + 15$ and $g(x) = 10x^2 + 2$.

II. SUPPLEMENTARY EXERCISES

1.

   a) Find the approximate solutions of the quadratic equation $3x^2 + 17x - 42 = 0$ by tracing the graph of $f(x) = 3x^2 + 17x - 42$.

   b) Where on the graph did you find these solutions?

2.

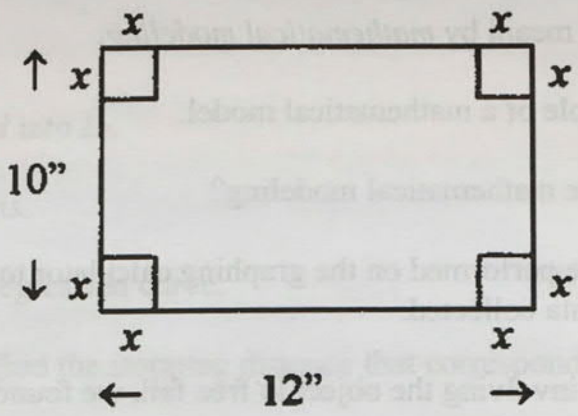
   a) Find the zero(s) of the function $f(x) = x^2 - 2x + 1$.

   b) How do we describe the position of the graph of $f(x)$ with respect to the x-axis?

3. Make up an equation of a quadratic function whose graph lies in quadrants I and II.

4. Make up an equation of a quadratic function whose graph is tangent to the x-axis and lies in quadrants III and IV.

5. An open-top box is made by cutting equal squares from the four corners of a flat 10×12-inch piece of cardboard and folding up the flaps:



a) Let x be the side-length of the four corner squares. Write an algebraic expression for the surface area of the open-top box.

b) What size squares should be removed to produce a box having a surface area of 104 square inches?

III. INVESTIGATIVE PROBLEM

1. A fence encloses a rectangular pen of 450 square feet. The pen's length equals twice its width.

a) Let x represent the width of the pen. Express the length in terms of x.

b) Write an expression for the area of the pen in terms of x.

c) How many feet of fence surround this pen? Specify the length and the width of this rectangle.

Lesson 17
Modeling Real-World Data with Quadratic Functions

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1.
a) Recall what is meant by *mathematical modeling*.

b) Give an example of a mathematical model.

c) Why do we use mathematical modeling?

2. Recall the steps we performed on the graphing calculator to find the regression equation that best fits the data collected.

3. In the exploration involving the object in free fall, we found $y = 506.47x^2 - 5.9x + 2.38$ for our regression equation.

a) Explain the meaning of x and y in this equation.

b) According to this model, what distance has the object fallen after 2.5 seconds?

4. In the exploration involving the small theater company we found $y = -.89x^2 + 23.43x + 12.76$.

a) Explain the meaning of x and y in this equation.

b) What would the theater's annual revenue be according to this model if it charged \$16.50 per ticket?

c) What ticket price would yield the maximum annual revenue?

II. SUPPLEMENTARY EXERCISES

1. The following table relates the speed (s) and the approximate stopping distance (d) of a certain vehicle.

s	d
0	0
5	0.6
10	1.8
15	2.5
20	4.2
25	5.6
30	6.8

a) Input s into L_1 and d into L_2.

b) Plot these data points.

c) Find the quadratic regression curve.

d) Using your model, find the stopping distance that corresponds to a speed of 60.

III. INVESTIGATIVE PROBLEM

1. Relating Area to Radius. Recall how we related circumference to diameter of a circle in Lesson 11. Here we will relate area to radius.

   a) Suppose you collected five discs and measured the radius and area of each as accurately as possible. Here are your hypothetical data:

Radius (r)	Area (A)
2.6	21
3.75	43.5
4.5	63.5
5.1	82.1
7	150.6

   b) Input the data as follows: the radius values in L_1 and the area values in L_2.

   c) Plot the data points and perform a quadratic regression. What model do you find?

   d) Use this model to find the area when $r = 10$. How does this compare with the actual value?

Lesson 18
Polynomial Explorations (Degree Greater than Two)

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. We have seen that a first-degree polynomial function is called *linear* and a second-degree, *quadratic*.

   a) What do we call a third-degree polynomial function? What is its standard form?

   b) What do we call a fourth-degree polynomial function? What is its standard form?

2. In exploration 1, we found that the volume of the open box constructed from the 16"×16" sheet of cardboard was given by the function $V(volume) = f(x) = 4x^3 - 64x^2 + 256x$, where x was the side length of the cut-out squares.

   a) Is $f(x)$ a cubic function? Is it logical to have a cubic function in this problem situation? Identify the constants a, b, c, and d.

   b) What volume would the box have if we cut out a 4"× 4" square from each corner?

3. At the end of exploration 1, you were asked to compute the third differences (D_3) and verify that they are constant (since we have a third-degree polynomial function). Here is your opportunity to do so:

x	$V = f(x)$	D_1	D_2	D_3
0	0			
1	196			
2	288			
3	300			
4	256			
5	180			
6	96			
7	28			
8	0			

   a) Compute the first differences (D_1).

   b) Compute the second differences (D_2).

   c) Compute the third differences (D_3).

d) What do you conclude?

4. In exploration 2, we found the following general expressions for an *nxnxn* cube:
   >> 8 for the number of unit cubes with 3 faces exposed
   >> $12(n-2)$ for the number of unit cubes with 2 faces exposed
   >> $6(n-2)^2$ for the number of unit cubes with 1 face exposed
   >> $(n-2)^3$ for the number of unit cubes with 0 faces exposed

   a) Can you explain the connection between the constants 8, 12, and 6 and the geometric structure of a cube?

   b) Show that $12(n-2)$, $6(n-2)^2$, and $(n-2)^3$ can be written in the standard form of a linear, quadratic, and cubic polynomial, respectively.

II. SUPPLEMENTARY EXERCISES

1. Suppose you have a rectangular box of length x.

   a) Its width is one-half its length. Express the width in terms of x.

   b) Its height is 0.4 times its length. Express the height in terms of x.

   c) Derive the algebraic equation for the volume function $V(x)$ in terms of x.

   d) Suppose the volume of the box is 200 cubic inches. Find the dimensions of the box (in inches).

III. INVESTIGATIVE PROBLEM

1. <u>Exploring Properties of Cubic Function Graphs</u>

   a) Graph $f(x) = x^3$, $g(x) = .2x^3 - 5x$, and $h(x) = x^3 + 2x^2 - 4x - 8$.

   b) Graph $-f(x)$, $-g(x)$, and $-h(x)$. ($-f(x) = -x^3$, $-g(x) = -.2x^3 + 5x$, and $-h(x) = -x^3 - 2x^2 + 4x + 8$).

   c) How many zeros (or roots) do $f(x)$, $g(x)$, and $h(x)$ have respectively? What might you conjecture about the number of zeros of a cubic function?

   d) What can you deduce about the general shape of a cubic function?

e) What do you notice about the number of y-intercepts? Can you prove your answer using the standard form $f(x) = ax^3 + bx^2 + cx + d$?

Lesson 19
Rational Functions—Introductory Explorations

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1.
   a) What is the general form of a rational function?

   b) What restrictions must be placed on the denominator? Why?

   c) What do the zeros of the numerator represent?

2. Give examples of a rational function that is the quotient of:

   a) a linear function and a quadratic function.

   b) a quadratic function and a cubic function.

   c) a cubic function and a linear function.

3. In exploration 1, we found the rational function $f(x) = \dfrac{2x^2 + 48}{x}$.

   a) What type of polynomial function is the numerator?

   b) What type of polynomial function is the denominator?

   c) In this functional relationship, what did x represent? What did $f(x)$ represent?

   d) What is the perimeter when the length is 7 linear units?

   e) What is the length when the perimeter is 30 linear units?

4. In exploration 1, we took a close look at the function $f(x) = 1/x$.

   a) What name do we call this function? Why?

   b) If we choose a very large x, what can you say about its reciprocal?

   c) If we choose a very small, positive x, what can you say about its reciprocal?

   d) If we choose a very large, negative x, what can you say about its reciprocal?

   e) What about the reciprocal of 0? How is this fact translated graphically?

II. SUPPLEMENTARY EXERCISES

1. In each of the following cases, state the value(s) of x for which the function is undefined:

   a) $f(x) = \dfrac{5}{x}$

   b) $f(x) = \dfrac{x}{x+5}$

   c) $f(x) = \dfrac{x^2+1}{x-7}$

   d) $f(x) = \dfrac{x^3}{1/2-x}$

2. The reciprocal of a number plus 4 equals 14.

   a) Write an algebraic expression for "the reciprocal of a number."

   b) Write an algebraic expression for "the reciprocal of a number plus 4."

   c) Suppose the above sum equals 14. What is the number? Explain how you found your answer.

III. INVESTIGATIVE PROBLEM

1. The XYZ car rental company charges a flat fee of \$30.00 plus \$0.25 per mile.

   a) Suppose you drove 200 miles in one day, what will be your cost?

   b) Suppose you drove x miles, write an algebraic expression for your cost.

   c) Write an algebraic expression for the average cost per mile for driving an XYZ car x miles. What type of a function is this?

   d) If your average cost per mile was \$0.31 how many miles did you drive?

Lesson 20
The Geometry of Rational Function Graphs

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. Exploration 1 was an in-depth analysis of the graph of $f(x) = \dfrac{1}{x-2}$.

a) Is $f(x)$ defined for $x = 2$? What is the graphical interpretation of this fact?

b) What is the y-intercept? How did you find it?

c) Are there any x-intercepts? Explain your answer.

d) Does the graph of $f(x)$ have a highest point?

e) Does the graph of $f(x)$ have a lowest point?

2. Now graph $-f(x) = -\dfrac{1}{x-2} = -f(x) = \dfrac{-1}{x-2}$ in the same xy-plane.

a) What is the relationship between the graph of $f(x)$ and that of $-f(x)$? Had you predicted this relationship?

b) What about the vertical asymptotes of these graphs?

c) How does the y-intercept of the graph of $f(x)$ compare with that of the graph of $-f(x)$?

d) What about the x-intercepts?

3. Without graphing, predict the position of the graphs of the following functions with respect to the graph of $f(x) = 1/x$:

a) $f(x) = \dfrac{1}{10+x}$

b) $f(x) = \dfrac{1}{x-\frac{1}{2}}$

c) $f(x) = \dfrac{-1}{x+5}$

II. SUPPLEMENTARY EXERCISES

In Lesson 20, you were asked to explore $h(x) = \dfrac{2}{x^2}$ and $k(x) = \dfrac{2x+4}{x^2-1}$. This is your opportunity to do so.

1. Graph $h(x) = \dfrac{2}{x^2}$. (Use these window limits: $x_{min} = -5$, $x_{max} = 5$, $y_{min} = -5$, $y_{max} = 5$.)

   a) Make two observations about this graph.

   b) Can you explain, algebraically, why this graph has no x or y-intercepts?

   c) Why is the entire graph of $h(x)$ above the x-axis?

   d) Graph $l(x) = \dfrac{12}{x^2}$, $m(x) = \dfrac{22}{x^2}$, and $n(x) = \dfrac{32}{x^2}$ alongside $h(x) = \dfrac{2}{x^2}$, using the same window limits. Make two observations. Can you generalize for $f(x) = \dfrac{a}{x^2}$, where a is any positive constant?

2. Graph $k(x) = \dfrac{2x+4}{x^2-1}$. [Use the following window limits: $x_{min} = -5$, $x_{max} = 5$, $y_{min} = -5$, $y_{max} = 5$. Do not forget the two sets of parentheses when you key in this function: $(2x + 4)/(x^2 - 1)$.]

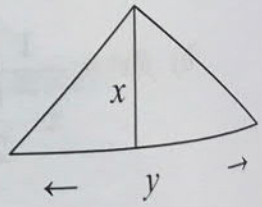
   a) Explain why the graph has <u>two</u> vertical asymptotes.

   b) How many sections is the graph "broken into"?

   c) Find the x- and y-intercepts, first by graphical exploration, then by symbol manipulation.

III. INVESTIGATIVE PROBLEM

1. Let x be the height and y the base of an arbitrary triangle, as shown.



   a) Write a formula for the area of the triangle.

   b) Suppose the area equals 5 square units. Write an equation expressing this fact.

c) Rewrite this equation expressing y in terms of x.

d) Suppose the height is 5, what is the base? Suppose the base is 1.25, what is the height?

Lesson 21
Working with Rational Functions and Equations

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1.
a) What did you learn about the graph of $f(x) = y = a/x$ when a is a positive constant?

b) What did you learn about the graph of $f(x) = y = a/x$ when a is a negative constant?

2. In problem 1, we derived the function $f(x) = y = 250/x$.

a) What did x represent?

b) What did y, or $f(x)$, represent?

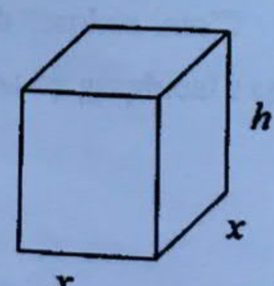
c) Suppose we wanted to express the average speed in terms of the time. [In other words, the time is the input variable (x) and the speed, the output variable (y).] Write the algebraic equation for the average speed in terms of the time.

d) If a vehicle traveled the 250-mile distance at an average speed of 55 mph, how long would it take? Is your answer consistent with what we found in the lesson?

e) If a vehicle traveled the same distance in 2 hours and 40 minutes, what was its average speed? Is your answer consistent with what we found in the lesson?

3. Problem 2 explored the minimum surface area of a rectangular, square-base box.

a) Let x denote the side-length of the base and h the height, as shown:



This rectangular box has six faces. Draw each one and express its corresponding area in terms of x and h.

b) Express the <u>total</u> surface area of the box.

c) How did we write this surface area in terms of x alone?

II. SUPPLEMENTARY EXERCISES

1. Revisiting the function $k(x) = \dfrac{2x+4}{x^2-1}$ from Lesson 20:

a) Graph $k(x)$ once again with these window limits: $x_{min} = -5$, $x_{max} = 5$; $y_{min} = -5$, $y_{max} = 5$.

b) For what values of x does the graph of $k(x)$ lie above the x-axis (or horizontal axis)?

c) For what value(s) of x does the graph of $k(x)$ intersect the x-axis?

d) For what values of x does the graph of $k(x)$ lie below the x-axis?

e) Examining the signs of $(2x + 4)$, $(x^2 - 1)$, and finally, $\dfrac{2x+4}{x^2-1}$, explain your answers to parts b), c), and d).

2. The following steps will guide you to solve this number problem:

The sum of twice a number plus twice the reciprocal of the number equals 5.

a) Write an algebraic expression for "twice a number."

b) Write an algebraic expression for "twice the reciprocal of the number."

c) Write an algebraic expression for the sum of the quantities in parts a) and b).

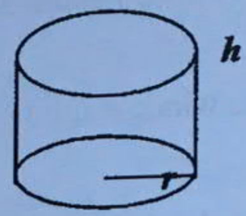
d) Write an algebraic equation for the entire problem situation.

e) Solve this problem by graphing the sum function S with the WINDOW limits: $x_{min} = -5$, $x_{max} = 5$; $y_{min} = -8$, $y_{max} = 8$. TRACE the graph and locate the points for which y (or S) equals 5. (The solutions are the corresponding x-values.)

f) Solve by using TBLSET (TblMin = 0, ΔTbl = .5), then TABLE: locate 5 in the y-column. (The solutions are the corresponding x-values.)

III. INVESTIGATIVE PROBLEM

1. A right cylindrical tin can must hold 250 ml of liquid. What dimensions (radius and height) would require the minimum amount of tin? (This is the same question we addressed in problem 2 of the videotape, except this time we have a cylinder.)



a) Write an algebraic expression for the surface area of the top and bottom of the can. (Draw these surfaces.)

b) Write an algebraic expression for the lateral surface area of the can. (Draw these surfaces.)

c) Write the algebraic equation for the total surface area S.

d) Express S as a function of r alone.

e) Minimize the function $S(r)$ with the aid of your graphing calculator.

Lesson 22
Exponential Functions—Introductory Explorations

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. In both explorations 1 and 2, we encountered the exponential function $f(x) = 2^x$.

a) What did 2^x represent in exploration 1?

b) What did 2^x represent in exploration 2?

2. We examined the graph of $f(x) = 2^x$.

a) Graph this function once again with the aid of your graphing calculator.

b) State five properties of this function.

3. Recall the mythical story of chess (exploration 1).

a) How many kernels of wheat are on square #5?

b) How many kernels of wheat are on square #10?

c) 10 is twice 5. Is your answer to b) twice your answer to a)? Why or why not?

4. Recall the paper-folding activity (exploration 2).

a) How high would the stack of papers be if you made only 30 folds? (Use the same fact: A stack of 300 sheets of paper is 1 inch high.)

b) After how many folds will the stack reach (or exceed) a height of 1 inch? 1 foot? 1 yard? 1 mile? (Once again, use 300 sheets = 1 inch.)

II. SUPPLEMENTARY EXERCISES

1. Explain why the following are not exponential functions.

a) $f(x) = y = x^3$

b) $f(x) = y = 3/x$

c) $f(x) = y = 5^2$

2. We saw that the sum $2^0 + 2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$. For example, $2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63 = 2^6 - 1$.

   a) Check if this relationship holds for powers of 3, 4, and 5.

   b) Verify the following relationship with the aid of your calculator: $(1/2)^0 + (1/2)^{-1} + (1/2)^{-2} + \ldots + (1/2)^{-n} = (1/2)^{-(n+1)} - 1$.

   c) Can you explain why this equation holds?

3. Graph $f(x) = 2^x$, $g(x) = 5^x$, $h(x) = (1/2)^x$, and $j(x) = (1/5)^x$ simultaneously with the following limits: $x_{min} = -5$, $x_{max} = 5$; $y_{min} = -2$, $y_{max} = 15$.

   a) What observations can you make about all four functions?

   b) Now, considering the pairs of functions $y = 2^x$ and $y = (1/2)^x$, or $y = 5^x$ and $y = (1/5)^x$, can you make additional observations?

III. INVESTIGATIVE PROBLEM

1. Summer Job Offers. Lesson 22 ended with a problem that consisted of choosing between two summer jobs for the month of August. Rewind the videotape, review the two propositions, and answer the following questions.

   a) Suppose the job offers were for the month of June. What difference would one day make?

   b) What would be your total earnings, in dollars, in each case (for the month of August)?

   c) Compare your earnings in the first half of the month with those in the second (for both jobs, in August).

   d) What type of functional relationships are these? Can you graph them?

Lesson 23
The Geometry of Exponential Function Graphs

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. Practice with Exponents. Write as a single power:

   a) $x^2 \bullet x =$

   b) $y^3 \bullet y^{-2} =$

   c) $(1/z)^3 \bullet z^3 =$

   d) $w^0 \bullet w^4 \bullet w^{-7} =$

   e) $v^3 \bullet (1/v)^{-2} \bullet v =$

2. We saw that the graphs of $f(x) = 2^x$ and $g(x) = (1/2)^x$ are symmetric with respect to the y-axis. Let us look at this property more closely.

   a) Evaluate $f(x) = 2^x$ for $x = 1$ and $g(x) = (1/2)^x$ for $x = -1$.

   b) Evaluate $f(x) = 2^x$ for $x = 2$ and $g(x) = (1/2)^x$ for $x = -2$.

   c) Evaluate $f(x) = 2^x$ for $x = 6$ and $g(x) = (1/2)^x$ for $x = -6$.

   d) Can you explain why these two functions' graphs are symmetric with respect to the y-axis?

3. Consider all exponential functions of the form $f(x) = b^x$ where b is a constant, such that $b > 0$ and b is not equal to 1.

   a) State two properties of the graphs of all these functions.

   b) Why do we eliminate the case where $b = 1$? Is the function $f(x) = 1^x$ undefined?

   c) Why do we eliminate the case where $b < 0$ (i.e., b is negative)?

II. SUPPLEMENTARY EXERCISES

1.
   a) Graph the functions $f(x) = 3^x$ and $g(x) = (1/3)^x$ with the following WINDOW limits: $x_{min} = -5$, $x_{max} = 5$; $y_{min} = -1$, $y_{max} = 15$.

   b) What is the relative position of the graphs of f and g with respect to the y-axis?

   c) What is the y-intercept of each graph?

   d) While we cannot see the graph of f as x becomes very small, what is its behavior?

   e) While we cannot see the graph of g as x becomes very large, what is its behavior?

2. Paramecia, like most protozoans, reproduce by dividing themselves into two cells. Suppose a laboratory experiment begins with one one-celled paramecium at 12 p.m., and suppose further that the division of cells occurs every hour, on the hour.

   a) How many paramecia will there be at 3:01 p.m.?

   b) How many paramecia will there be at 10:01 p.m.?

   c) Write a function f that gives the number of paramecia, $f(x)$, x hours beyond 12 p.m. (for more precision, suppose it is 1 minute after x hours).

3. A sheet of paper has an area of 1 square foot. After folding the paper in half, two regions are formed, each with an area of 1/2 sq. ft. Suppose we continued the process of "folding in half."

   a) What is the area of the smallest region formed after two folds? Three folds?

   b) What is the area of the smallest region formed after six folds?

   c) Write a function f that gives the area of the smallest region $f(x)$ after x folds.

   d) Graph the function you found in part c).

III. INVESTIGATIVE PROBLEM

1. In Lesson 23, we studied functions of the form $f(x) = b^x$ ($b > 0$ and $b \neq 1$). We note that the exponent x is a linear function in x of the form $ax + b$, where $a = 1$ and $b = 0$. There are many exponential functions of the form $f(x) = b^{ax+b}$, where the variable still lies in the exponent position, but this exponent is more elaborate. The following function is such an

example: $f(x) = 10^{.00389x+2}$ gives the approximate world population, $f(x)$, as a function of the year x.

a) Show that the exponent alone is a linear expression of the form $ax + b$.

b) Graph the function $f(x)$ with the following WINDOW limits: $x_{min} = 1700$, $x_{max} = 2010$, $y_{min} = 0$, $y_{max} = 6{,}000{,}000{,}000$ (6 billion). Do not forget the parentheses around the entire exponent when keying in $f(x)$: 10 ^ (.00389x + 2).

c) What was the world population in 1900? In 1980?

d) When will the world population reach 6 billion people?

Lesson 24
Working with Exponential Functions and Equations

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. In the radioactive decay problem, we found $f(x) = 23.03(.67)^x$ for our regression equation.

   a) What did 23.03 represent?

   b) If x were measured in 1000-year intervals, we said that this function could approximately model the behavior of radium. What was its half-life according to this model?

   c) The quarter-life is the time required for the substance to diminish to one fourth of the original amount. Find the quarter-life according to this model.

2. In the 800-meter run problem, we found the following regression equations:

   $y_1 = 113.748(.986)^x$ for the men's world records
   $y_2 = 153.925(.960)^x$ for the women's world records

   a) What year corresponded to $x = 0$? What period of time corresponded to a unit increment on the x-axis (horizontal axis)?

   b) Graph y_1 and y_2 with the following WINDOW limits: $x_{min} = 0$, $x_{max} = 20$, $y_{min} = 50$, $y_{max} = 160$.

   c) Find the coordinates of the intersection point of both graphs. What does this point represent?

   d) According to these models, what were the respective world records in 1995? Check the accuracy of this information.

   e) Use these models to predict the respective world records in 2005. Do they seem reasonable?

II. SUPPLEMENTARY EXERCISES

1. Graph the functions $f(x) = (2.5)^x$, $g(x) = 3(2.5)^x$, and $h(x) = 7.5(2.5)^x$ with the following WINDOW limits: $x_{min} = -5$, $x_{max} = 5$, $y_{min} = -1$, $y_{max} = 10$.

a) Are these functions of the form $f(x) = Ab^x$? If so, identify the values of the constants A and b in each case.

b) In which quadrant(s) lie the graphs of functions f, g, and h? If these functions modeled real-world situations, which portion of their graphs would have meaning?

c) In real-world situations, would these functions model exponential growth or exponential decay? Explain your answer.

d) Give the y-intercept of each graph. In real-world situations, what would these points represent?

2. Graph the functions $f(x) = (0.4)^x$, $g(x) = 4.5(0.4)^x$, and $h(x) = 9(0.4)^x$ with the same WINDOW limits as in II.1. Answer the same four questions, a) through d), as you did in problem II.1.

3. Do you now better understand the general form $f(x) = Ab^x$?

   a) Why is this called an exponential function?

   b) Show that the form $f(x) = b^x$ is a special case of the general form $f(x) = Ab^x$.

   c) Summarize the effect of A on the graph of $f(x)$. Explain why A is positive ($A > 0$) in most real-world situations.

   d) Summarize the effect of b on the graph of $f(x)$.

III. INVESTIGATIVE PROBLEM

1. A city has a population of 70,000 that is increasing at a rate of 3.5% per year.

   a) Show that the function $f(x) = 70{,}000(1.035)^x$ gives the population of the city x years from now.

   b) Is this population function of the form $f(x) = Ab^x$? Explain your answer.

   c) Let $x = 0$ represent the start of the year 1997. Predict the population at the start of the year 2005?

   d) In what year will the population reach 100,000 people? [Graph $f(x)$ with the following window limits: $x_{min} = 0$, $x_{max} = 25$, $y_{min} = 70{,}000$, $y_{max} = 160{,}000$.]

e) When will the population of this city double?

Lesson 25
Systems of Linear Functions and Equations

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. For each of the following systems of two linear equations in two variables (or two unknowns), rewrite the system in the form:

$$ax + by = c$$
$$a^i x + b^i y = c^i$$

a) $y = -x + 5$
$y = x - 3$

b) $y = (1/2)x + 2/3$
$y = (-3/2)x + 1/4$

c) $y = 1.7x + 10$
$y = -10x - 1.7$

2. In problem 1, we found the following system:

$$s + c = 30 \quad (I)$$
$$3s + 4c = 100 \quad (II)$$

where s represented the number of stools and c, the number of chairs.

a) Which equation referred to the total number of seats? Which equation referred to the total number of legs?

b) Suppose the next day, Sam and his carpenter built 35 seats, using a total of 122 legs. Write the system of equations that models this new situation.

c) Solve the system using the <u>substitution</u> method.

d) Solve the system using the <u>elimination</u> method.

e) Solve the system through <u>graphical exploration</u>.

f) Check your answer.

3.

a) When are two linear equations in two unknowns said to be <u>inconsistent</u>? What is the relative position of their respective graphs?

b) Same question for <u>consistent</u> equations.

c) Same question for linearly dependent equations.

II. SUPPLEMENTARY EXERCISES

1. Solve the following system using the elimination method.

$$2x - 3y = 4$$
$$5x + 4y = 7$$

2. Solve the following using the substitution method.

$$8x - 10y = 0$$
$$-12x + 15y = 60$$

3. Solve the following system using your graphing calculator.

$$3x - 4y = 5$$
$$12y - 9x + 15 = 0$$

4. Solve the following system using the method of your choice.

$$2/x - 3/y = 4$$
$$1/x + 4/y = -10$$

III. INVESTIGATIVE PROBLEM

1. A customer in a coffee house wants to buy a blend of two kinds of coffee: Colombian coffee, selling for \$8.00 per pound, and Brazilian coffee, selling for \$7.50 per pound.

   a) He buys 4 pounds of the blend. Let c and b be the number of pounds he bought of Colombian and Brazilian coffee, respectively. Write an equation for the total number of pounds of coffee he bought.

   b) He spent \$30.75 for the 4-pound blend. Write an equation for his cost in terms of b and c.

   c) How many pounds of each kind of coffee did he buy?

   d) Check your answer.

Lesson 26
Using Matrices to Solve Linear Systems

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. Problem 1 yielded the following system of two linear equations in two unknowns:

$$x + y = 120$$
$$3.75x + 1.25y = 307.50$$

a) Give the coefficient matrix A.

b) Give the variable matrix X.

c) Give the constant matrix B.

d) Write a matrix equation with A, X, and B that is equivalent to the above system.

2. The solution to problem 1 was 63 servings of pizza and 57 servings of cake.

a) Verify that these answers are correct.

b) Use the graphical exploration you learned in Lesson 25 to show that the system in problem 1 is consistent. (Solve both equations for y, the number of cake servings, and graph them with the following WINDOW limits: $x_{min} = y_{min} = 0$, $x_{max} = 125$, $y_{max} = 250$.)

3. The dimension (or size) of a matrix is said to be mxn, where m is the number of rows and n is the number of columns.

a) Give the dimensions of matrices A, X, and B of problem 1.

b) Give the dimensions of matrices A, X, and B of problem 2.

II. SUPPLEMENTARY PROBLEMS

1. Give the dimensions of the following matrices:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -4 & -3 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 1/2 & 2/3 \\ -5 & 10 \end{bmatrix} \qquad C = \begin{bmatrix} 1.5 & 2 \end{bmatrix}$$

2. A <u>square</u> matrix is such that $m = n$; we call m or n the <u>order</u> of the square matrix.

Let A and B be two square matrices of order 2:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}.$$

The sum and product matrices, $A + B$ and AB, are defined as follows:

$$A + B = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix};\ AB = \begin{bmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{bmatrix}.$$

a) Compute $A+B$ and AB, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -2 \\ -3 & 5 \end{bmatrix}$

b) Compute $B + A$ and BA.

c) What might you conjecture regarding the commutative property of matrix addition and multiplication?

3. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 10 \end{bmatrix}$.

a) Find A^{-1} with the aid of your graphing calculator.

b) Compute AA^{-1} using the definition given in problem II.2. Then compute $A^{-1}A$.

c) Compute AA^{-1} and $A^{-1}A$ using your graphing calculator.

d) Deduce the 232 identity matrix, I, for matrix multiplication.

e) Show that $AI = IA = A$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

f) Can you deduce the 3×3 identity matrix for matrix multiplication?

III. INVESTIGATIVE PROBLEMS

For each of the following problems, write a system of linear equations and solve using matrices and your graphing calculator.

1. Suppose your drama class sold 500 tickets to the class play and grossed \$2,900. Adult tickets sold at \$7.00 each and student tickets at \$4.00 each. How many of each were sold? (Let a = # of adult tickets sold; s = # of student tickets sold.)

2. A collection of nickels, dimes, and quarters is worth \$11.25. There are twice as many dimes as nickels, and there are 95 coins in all. How many of each type of coin are in this collection? (Let n, d, q denote the number of nickels, dimes, and quarters, respectively.)

Lesson 27
Systems of Functions and Equations

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. Suppose you had to solve a system of the form:

$$y = ax + b$$
$$y = a'x^2 + b'x + c'$$

Discuss the possible solution sets.

2. Same question as #1 for the system:

$$y = ax + b$$
$$y = a'/x$$

3. In problem 1, explain why we discarded the solution $x = -\sqrt{7}$.

4. Suppose that the area of the rectangle in problem 4 were 70 square units.

a) Write the system of equations that models this new situation.

b) What are the new dimensions of the rectangle?

c) Check your answers.

5. The last problem in Lesson 27 explored the solutions to the system:

$$y = x^2$$
$$y = 2^x$$

a) Recall the solutions.

b) Explain the meaning of these ordered pairs in terms of the graphs of $y = x^2$ and $y = 2^x$.

c) Check the solutions algebraically with the aid of your calculator.

II. SUPPLEMENTARY EXERCISES

1. You were asked to solve problem 5 on your own. For your convenience, here is the statement of the problem:

   The sum of two numbers is 30, and their product is 221. What is the positive difference between the two numbers?

   a) Write a system of equations that models this situation.

   b) Solve the system by examining the graphs of these equations and finding their point(s) of intersection with the aid of your graphing calculator.

   c) Solve the system using TBLSET and TABLE on your graphing calculator.

2. Find the solution sets of the following systems of functions/equations through functional exploration (using GRAPH and CALC/intersect or TBLSET and TABLE). Verify your solutions through symbol manipulation whenever possible.

   a) $y = 2x^2 + 5$
   $y = 2x^2 + 3x - 4$

   b) $y = -x^2 + 5$
   $x - y + 1 = 0$

   c) $xy = 5$
   $x + y = 0$

III. INVESTIGATIVE PROBLEM

1. Consider the system of functions: $f(x) = x^3$
   $g(x) = 3^x$

   a) In what quadrant(s) lies the graph of $f(x)$? Graph $f(x)$ to confirm your answer.

   b) In what quadrant(s) lies the graph of $g(x)$? Graph $g(x)$ to confirm your answer.

   c) How many solutions do you predict? In what quadrants will the intersection points lie?

   d) Find the solution set to this system of functions.

Lesson 28
Systems of Inequalities

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. Suppose a solution set exists for each of the following inequalities. Describe the general form of the solution set in each case:

a) One linear inequality in one unknown.

b) One linear inequality in two unknowns.

c) One system of two linear inequalities in two unknowns.

2. In problem 2, the solution set of the inequality $-3x + 4 \leq 2$ was $x \geq 2/3$. Compare this solution set with the solution set of $-3x + 4 < 2$.

3. In problem 3, the solution set of the inequality $2x - y < 3$ was the half-plane lying "above" the line $y = 2x - 3$, not including the points on the line. Compare this solution set with the solution set of $2x - y \leq 3$.

4. In problem 4, we solved the system
$$\begin{aligned} -2x + 3y &> -4 \\ x + y &< 3 \end{aligned}$$

a) Give the equations of the two lines we graphed.

b) These lines divided the xy-plane in how many regions?

c) The solution set of the system was the region of points (x,y) that includes $(0,0)$. Verify this fact algebraically.

II. SUPPLEMENTARY PROBLEMS

1. Solve the following inequalities and sketch their respective solution sets.

a) $2x + 1/2 < (1/2)x - 2$

b) $-0.5x + 7 \geq 3.2x$

c) $-3x + y \leq 4y + 5x$

d) $-y + 5x + 2 > 0$

2. Solve the following systems of inequalities and sketch their respective solution sets.

a) $y < x$
$2y - 2x < 3$

b) $(1/2)x + y < 5$
$y > 2x + 5$

c) $y \leq x^2 + 4$
$y \geq 2$

d) $y > x^2 - x - 2$
$y < (3/2)x + 1$

III. INVESTIGATIVE PROBLEM

1. At the end of the lesson, I cautioned you against "one-way mathematics" and encouraged you to practice starting with a solution set and ending with the system of inequalities that yields the given solution set. That is the object of this investigative problem.

Suppose the solution set of a system of linear inequalities is graphically represented by the following region:

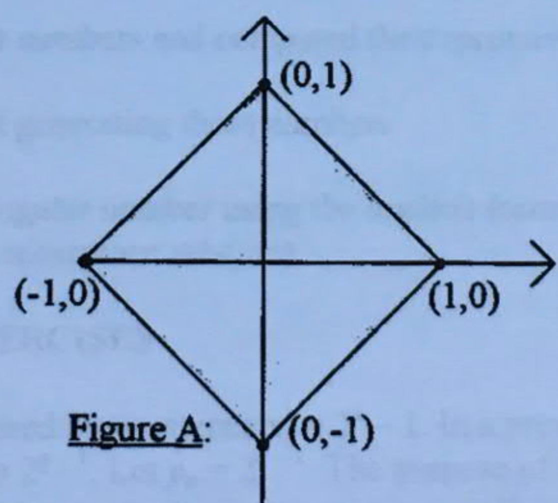


__Figure A__:

Let us find the system of inequalities that yields this solution set.

a) Find the equation of the line joining the points (0,1) and (1,0). Write an inequality whose solution set is graphically represented by the region of points lying __on or below__ this line.

b) Find the equation of the line joining the points (0,–1) and (1,0). Write an inequality whose solution set is graphically represented by the region of points lying on or above this line.

c) Find the equation of the line joining the points (–1,0) and (0,–1). Write an inequality whose solution set is graphically represented by the region of points lying on or above this line.

d) Find the equation of the line joining the points (–1,0) and (0,1). Write an inequality whose solution set is graphically represented by the region of points lying on or below this line.

e) Give the system of inequalities whose solution set is represented by Figure A.

Lesson 29
Iterating Functions—Looking at Functions Recursively

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. In this lesson, we mentioned a variety of instances of "everyday recursion." Can you find others?

2. In this lesson, we iterated the recurrence relation $x_n = 2x_{n-1} + 1$, starting with $x_1 = 1$.

a) Write this same recurrence relation using function notation.

b) Suppose we wished to iterate $x_n = (1/2)x_{n-1} + 5$ starting with $x_1 = 5$. By hand or with the help of your graphing calculator, find x_5 and x_{10}. What do you notice?

c) Often, the initial condition (or initial/starting value) is x_0 instead of x_1. Compute x_5 for the recurrence relation $x_n = 2x_{n-1} - n$, where $x_0 = 3$.

3.

a) Explain the difference between calculating the seventh Fibonacci number using the recurrence relation and the explicit formula.

b) Practice using the explicit formula by calculating F_3, F_4, F_5. (Use your calculator.)

4. We revisited the triangular numbers and compared their recursive and explicit definitions.

a) Recall the two ways of generating these numbers.

b) Compute the sixth triangular number using the explicit formula first, then the recursive one (i.e., the recurrence relation).

II. SUPPLEMENTARY EXERCISES

1. In this lesson, we encountered the sequence $x_n = 2^n - 1$. In a previous lesson, we encountered the expression 2^{n-1}. Let $y_n = 2^{n-1}$. The purpose of this exercise is to appreciate the difference between the two expressions $x_n = 2^n - 1$ and $y_n = 2^{n-1}$.

a) Explain in your own words, the algebraic difference between the quantities $2^n - 1$ and 2^{n-1}.

b) For $n = -2, -1, 0, 1, 2, 3$, evaluate $x_n - 2^n - 1$ and $y_n - 2^{n-1}$. How do they compare?

c) Graph the functions $f(x) = 2^x - 1$ and $g(x) = 2^{x-1}$ using your graphing calculator. Do the graphs confirm your answers to part b)?

2. Using the method of your choice, evaluate the first 12 Fibonacci numbers, F_1 through F_{12}.

   a) Compute the successive ratios F_2/F_1, F_3/F_2, F_4/F_3, … F_{12}/F_{11} using your calculator. What observations can you make?

   b) What can you conjecture about the value of F_n/F_{n-1} as n becomes very large?

3. We recursively define $x_n = x_{n-1} - x_{n-2} + x_{n-3}$ for $n \geq 3$, where $x_0 = 0$, $x_1 = 1$, $x_2 = 2$.

   a) Why is this recurrence relation called a third-order recurrence relation?

   b) Calculate the first few terms of the sequence until the pattern is clear. What set of values makes up this sequence?

4. Consider from a recursive point of view, the total number of line segments, l_n, needed to connect n given points that are not collinear (i.e., that are not in a straight line).

   a) Calculate l_1 through l_5 indicating the new line segments needed when going from l_{n-1} to l_n. (Make a picture.)

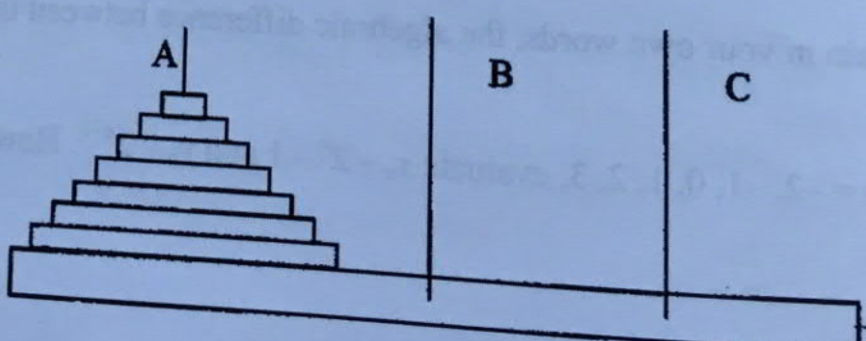
   b) Generalize by writing a recurrence relation for l_n.

   c) Can you find an explicit formula for l_n?

III. INVESTIGATIVE PROBLEM

1. The Tower of Hanoi. The object of this ancient game is to move the tower of seven disks (of graduated size) from peg A to peg C in the fewest possible moves, following these rules:

   (1) You may move only one disk at a time.
   (2) You may not place a disk on top of one that is smaller.



a) What is the minimum number of moves required for 1 disk? 2 disks? 3 disks? 4 disks?

b) Write a recurrence relation for the minimum number of moves, M_n, required to move n disks.

c) Can you find an explicit formula for M_n?

Lesson 30
Using Iteration as a Problem-Solving Tool

I. VIDEOTAPE FOLLOW-UP QUESTIONS

1. In exploration 1, we obtained the Sierpinski Triangle (ST) in two different ways.

   a) Briefly explain the two approaches.

   b) Draw the second iteration of the ST using the geometric approach.

2. Consider the geometric approach to the ST:

   a) Give the first five elements, x_0 through x_4, of the sequence x_n, where x_n denotes the number of triangles remaining in the ST after n iterations.

   b) Give the explicit formula (or rule) for x_n. (Express your answer in both sequence and function notation.)

   c) Define x_n recursively. (Again, give your answer in both notations.)

3. In exploration 2 we explored the concept of simple annual interest. Suppose you invested \$5,000 at 8% simple annual interest instead of \$1,000 at 6%.

   a) Let A_n be the amount of money in your bank after n years. Compute A_{10} and A_{20}.

   b) Define A_n explicitly. (Express your answer in both sequence and function notation.)

   c) Define A_n recursively. (Again, give your answers in both notations.)

4. In exploration 3, in order to compute the final balance after 4 years, we iterated the following recurrence relation 48 times on the graphing calculator:

   $ANS\ (1 + .085/12) - X$, where 14,500 was the initial value and X, the monthly payment.

   a) Compute the final balance after 4 years for $X = 350$. What is the significance of your answer?

   b) We found $y = -56.93x + 20{,}347.34$ to be the regression line for this problem and $r = -1$, the correlation coefficient. Explain the meaning of $r = -1$.

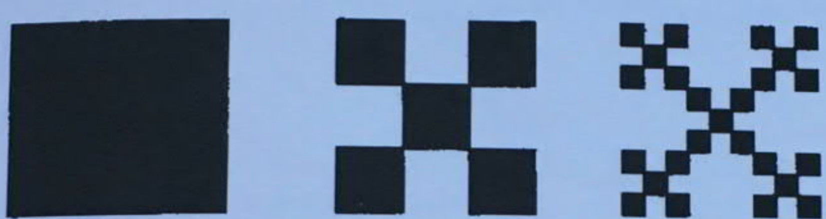
c) What variable is measured on the x-axis (the input axis)? What variable is measured on the y-axis (the output axis)?

d) What is the meaning of the x-intercept we found, (357.40, 0)? What is the meaning of the y-intercept we found, (0, 20, 347.34)?

e) How much money did you end up paying for your $15,000 car?

II. SUPPLEMENTARY EXERCISE

1. The following is the geometric construction of another fractal called the Box Fractal (BF).



$n = 0$ $\quad$ $n = 1$ 1st iteration $\quad$ $n = 2$ 2nd iteration

As with the triangles in the ST, the squares in the BF increase in number and decrease in size as the number of iterations increases.

a) Describe the geometric iterative process that generates the BF.

b) How many squares are there in the 4th iteration of the BF?

c) Let x_n denote the number of squares in the BF after n iterations. Define x_n explicitly and recursively. (Use the notation of your choice.)

III. INVESTIGATIVE PROBLEM

1. Suppose you invest $3,000 in a bank that pays 7.5% interest on your money, compounded monthly.

a) How much money would you have in your account after 5 years? After 10 years?

b) Now suppose that each month you deposit $100 into your account, in addition to the interest that accrues. Write a recurrence relation for A_n, the amount of money in the bank after n months.

c) Compute A_{60} and A_{120} using the iteration feature on your graphing calculator. Continuing to add \$100 to your account each month, how long would you have to wait for your investment to be worth \$30,000 (or 10 times the initial principal)?

ANSWERS

Lesson 1
An Overview

I.

1.

a) 1
b) x
$x + 4$
$3(x + 4) = 3x + 12$ (The distributive law—see Lesson 3.)
$3x + 12 - 9 = 3x + 3$
$2(3x + 3) = 6x + 6$
$(6x + 6)/6 = (6x)/6 + 6/6 = x + 1$
$x + 1 - x = 1$

2. Answers will vary.

3. Example:

7	8	9
14	15	16
21	22	23

a) The sum of the 9 numbers is 135.
b) $135/15 = 9$
c) Let n be the central number. We can write the 3×3 array as follows:

$n - 8$	$n - 7$	$n - 6$
$n - 1$	n	$n + 1$
$n + 6$	$n + 7$	$n + 8$

The sum of the 9 numbers is: $(n - 8) + (n - 7) + (n - 6) + (n - 1) + n + (n + 1) + (n + 6) + (n + 7) + (n + 8) = [(n - 8) + (n + 8)] + [(n + 7) + (n - 7)] + [(n - 6) + (n + 6)] + [(n - 1) + (n + 1)] + n = 2n + 2n + 2n + 2n + n = 9n$

Dividing $9n$ by n gives 9!

II.

1.

a) x

b) Yes.

c) x

$x + 1$

$9(x + 1) = 9x + 9$

$9x + 9 + x = 10x + 9$

$10x + 9 - 4 = 10x + 5$

Deleting the ones digit means deleting the digit 5. We are left with $10x$, which means that the original number has "moved" to the tens place.

2.

a) $\frac{1}{x}+\frac{1}{y}=\frac{y+x}{xy}$

b) $\frac{x+y}{w+z}=\frac{x}{w+z}+\frac{y}{w+z}$

c) $(a + b)^2 = a^2 + 2ab + b^2$ (See Lesson 3.)

d) $x^2x^3 = x^5$

e) $a^{-1} = 1/a$

f) $(-x) + (-x) = -2x$

III.

1. The error arises when both sides of the equation are divided by $(x - y)$:

This factor is zero since $x = y$, and dividing by 0 is an error because it gives an undefined (nonsensical) quotient.

Lesson 2
The Evolution of Numbers

I.

1. Review the videotape.

2.

a) 10
b) 500
c) 100
d) 50
e) 1000
f) 1
g) 5
h) 155
i) 1503
j) 18

3.

a) $\mathbf{N} = \{1, 2, 3, 4, \ldots +\infty\}$
b) The set of integers. This set is denoted by **Z**.
c) $\mathbf{Q} = \{a/b \text{ such that } a,b \in \mathbf{Z},\ b \neq 0\}$
d) The set of real numbers. This set is denoted by **R**.

4.

a) Yes. The solution is 4 and $4 \in \mathbf{N}$.
b) No. The solution is –2 and $-2 \notin \mathbf{N}$.
c) No. The solution is 1/2 and $1/2 \notin \mathbf{Z}$.
d) Yes. The solution is 1/3 and $1/3 \in \mathbf{Q}$.
e) No. The solution is $\pm\sqrt{3}$ and $\pm\sqrt{3} \notin \mathbf{Q}$.
f) Yes. The solution is $\pm\sqrt{5}$ and $\pm\sqrt{5} \in \mathbf{R}$.

II.

1. Use your calculator.

a) $0.\overline{6}$ (repeating)
b) -0.625 (terminating)
c) $0.\overline{18}$ (repeating)
d) $1.\overline{428571}$ (repeating)

e) -9.375 (terminating)
f) $0.\overline{370}$ (repeating)

2. Use your calculator.
   a) 3.141592654…
   b) 2.718281828… (evaluate $e^1 = e$)
   c) 1.414213562…
   d) –2.645751311…
   e) 5.196152423…

3.
   a) Yes.
   b) No. For example: $5 - 2 \neq 2 - 5$ and $5/2 \neq 2/5$.

4.
   a) Yes.
   b) No. For example: $2 - (3 - 4) \neq (2 - 3) - 4$; $2 - (3 - 4) = 3$ and $(2 - 3) - 4 = -5$. Likewise, $20 \div (10 \div 5) \neq (20 \div 10) \div 5$; $20 \div (10 \div 5) = 10$ and $(20 \div 10) \div 5 = 0.4$.

5.
   a) Yes.
   b) Using the distributive law:
      (i) $3(4 - 3) = 3(4) - 3(3) = 12 - 9 = 3$
      (ii) $-5(5 + 10) = (-5)(5) + (-5)(10) = -25 - 50 = -75$
      (iii) $13(12 - 8) = 13(12) - 13(8) = 156 - 104 = 52$

      Using another method:
      (i) $3(4 - 3) = 3(1) = 3$
      (ii) $-5(5 + 10) = -5(15) = -75$
      (iii) $13(12 - 8) = 13(4) = 52$

III.

1.
   a) Yes, 0 is the additive identity in **R** because $x + 0 = 0 + x = x$ for all real numbers x. (It is called the *additive identity* because any other number added to 0 remains unchanged or *identical* to itself.)

   b) Yes, 1 is the multiplicative identity in **R** because $x(1) = 1(x) = x$ for all real numbers x. (It is called the *multiplicative identity* because any number multiplied by 1 remains unchanged or *identical* to itself.)

   c) Yes. Every real number x has an additive inverse $(-x)$ such that $x + (-x) = (-x) + x = 0$. Therefore x' (x prime) $= -x$.

d) No, not "every" real number x. We must specify as follows: Every real number <u>except</u> 0 has a multiplicative inverse $(1/x)$ such that $x(1/x) = (1/x)x = 1$. Therefore, $x'' = 1/x$.

Lesson 3
The Language of Algebra

I.

1.

a) $a = -3, n = 0$
b) $a = 1, n = 1$
c) $a = 2, n = 1$
d) $a = 1/3, n = 1$
e) $a = -4/5, n = 2$
f) $a = 1, n = 3$
g) $a = -1.5, n = 4$
h) $a = -1, n = 5$

2.

a) $-2x$. A monomial, because the three expressions are like terms.
b) $3x^2$. A monomial, because the three expressions are like terms.
c) $-2x^2 + 4x + 6$. A polynomial, because the three expressions are not like terms.
d) $x^2 + 2xy + y^2$. A polynomial, because the three expressions are not like terms.

3.

a) x^2
b) $6x^5$
c) $28y^5$
d) $-2z^6$

4.

a)	$2x$	A monomial: $a = 2, n = 1$
b)	$1/2$ or 0.5	A monomial: $a = 1/2, n = 0$
c)	$(3/2)x$ or $1.5x$	A monomial: $a = 3/2, n = 1$
d)	$(1/5)x^{-2}$ or $1/(5x^2)$	Not a monomial because $n = -2$

II.

1.

a) $-x^2 - x + 1$
b) $-3x^2 + 9x - 3$
c) $8x^3 - 32x$
d) $6x^2 + 17x + 7$
e) $9x^2 + 6x + 1$
f) $16 - x^2$

2. a and 3, b and 2, c and 5, d and 7, e and 1, f and 9, g and 4, h and 8, i and 6

3.

a) $2x(2 - x)$
b) $2xy(2y + 3x)$
c) $(4x - 2)(4x + 2)$
d) $(2x + 5)^2$
e) $(2x - 1)^2$

III.

1.

a) $2[(2x) + (x + 5)] = 2(3x + 5) = 6x + 10$
b) $2x(x + 5) = 2x^2 + 10x$
c) $(2x) + (x + 5) = 3x + 5$
d) $6x + 10 = 70$
e) $2x^2 + 10x = 300$

Lesson 4
Exploring Functions with the Aid of Graphing Calculators

I.

1.

Equations to be solved:

b) Solution: $y = 9$

e) Solution: $x = 1/2$

Formulas:

c) Perimeter of a rectangle

f) Area of a circle

Identity:

d) The difference of two squares. (Holds for all quantities x and y.)

Property:

a) The commutative property of addition. (Holds for all real numbers x and y.)

2. 5.25

3. $(4 + 2t)(4 - 2t) = 16 - 4t^2$

4. $2(1/2) = 1$; $10(1/10) = 1$; $2/3(3/2) = 1$; $(-1/5)(-5) = 1$

5.

a) $y = 35.75$ (dollars) $[5.50 \times 6.5 = 35.75]$

b) $x = 10$ (hours) $[5.50 \times 10 = 55]$

II.

1.

a) On the x-axis, all points have a y-coordinate equal to zero, namely $y = 0$.

b) $y = 3$, because all points on this horizontal line have a y-coordinate equal to 3.

2.

a) On the y-axis, all the points have an x-coordinate equal to zero, namely $x = 0$.

b) $x = -2$, because all points on this vertical line have an x-coordinate equal to -2.

3.

c) The slope resembles the section of a bowl, except that the two "branches" continue upward to positive infinity.

This graph is made up of all the points (x,y) in the xy-plane whose coordinates verify the equation $y = x^2$; for example, $(-2,4)$, $(-1,1)$, $(0,0)$, $(3,9)$, and so on.

d) A number very close to 9 (9 is the exact value). A number very close to 4.47.

e) $y = 6.25$; $x = \pm 1.5$

III.

1.

d) The variables x and y seem to be related by a linear function (a function whose graph is a straight line. See Section II.)

Lesson 5
Linear Functions—Introductory Explorations

I.

2. 86
3. 88th
4. 51
5. 50th
6. \$138.75
7. 5 hours
8. \$70
9. 7 payments
10. As the electrician's number of hours of work increases, so does his pay. As the number of payments increases, the balance (the amount you still owe) decreases.
11. $f(x) = y = ax + b$, where a and b are constants.

II.

1.

a) 1, 3, 5, 7
b) $2n - 1$
c) Yes. It is of the form $ax + b$, where $a = 2$ and $b = -1$. Also, the points (1,1), (2,3), (3,5), and (4,7) live on a line.

2.

a) 1, 4, 7, 10
b) $3n - 2$
c) Yes.

a) 1, 5, 9, 13
b) $4n - 3$
c) Yes.

3. $2n - 1$, $3n - 2$, $4n - 3$. The leading coefficients (2,3,4) increase by one, and the constants (–1,–2,–3) decrease by one. This has to do with the geometric shapes: two, three, and four branches (of equal length), respectively, and one in the center.

III.

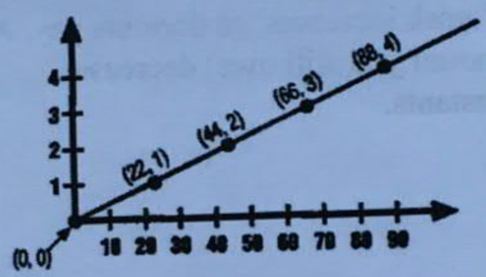
1.

a)

m	g
0	0
22	1
44	2
66	3
88	4
m	$m/22$ or $1/22m$

b) $g = (1/22)m$ (of the form $g = \square \cdot m$)

c)



d) about 11.5 gallons of gas

e) 330 miles

Lesson 6
Multiple Representations of Linear Functions

I.

1.
a) 25
b) 25
c) Any two points yield the same slope because the slope of a line is unique no matter which two points are selected.

2.
a) –18
b) –18
c) See I.1.c).

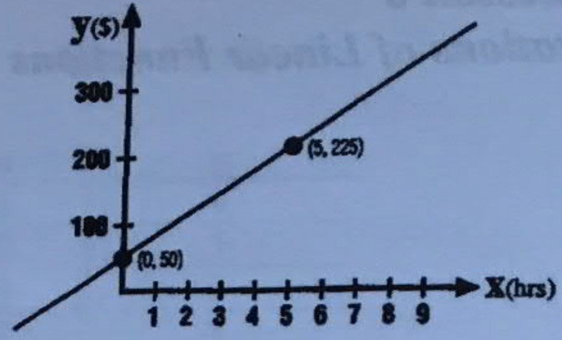
3. As the number of hours increases, so does your cost for the electrician's work. As the number of reimbursement payments increases, your outstanding balance decreases (until you pay off your loan entirely).

4.

Hours (x)	Dollars (y)
0	45
0.5	57.5
1	70
1.5	82.5
2	95
2.5	107.5
3	120
3.5	132.5
4	145
4.5	157.5
5	170

5. $C = 35h + 50$ or $y = 35x + 50$

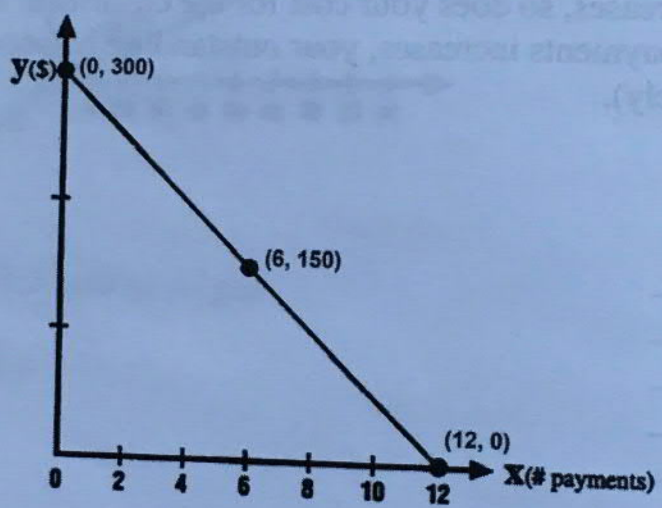
6.



a) 35
b) (0,50)
c) The cost of the home visit alone.

7. $B = -25n + 300$ or $y = -25x + 300$

8.



a) –25
b) (0,300)
c) At the start of the repayment plan (before any payments are made, $x = 0$), the outstanding balance is \$300 ($y = 300$).
d) (12,0) means that after 12 equal payments of \$25 ($x = 12$), the final balance is \$0 ($y = 0$); i.e., your loan is paid off.

II.

1.

a) $\dfrac{15-3}{10-0} = \dfrac{12}{10} = \dfrac{6}{5} = 1.2$
b) (0,3)

c) $y = 6/5x + 3$ or $y = 1.2x + 3$

d)

Input (x)	Output (y)
–4	–1.8
–3	–0.6
–2	0.6
–1	1.8
0	3
1	4.2
2	5.4
3	6.6
4	7.8

3.

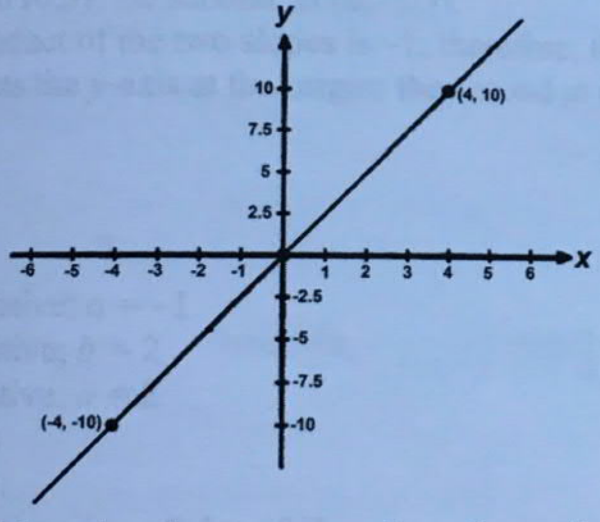
a) 2.5

b) (0,0)

c) $f(x) = y = 2.5x$

d) Suppose your favorite health bar costs $2.50. $y = 2.5x$ expresses the cost (y), in dollars, as a linear function of the number of bars (x) that you purchase.

e)



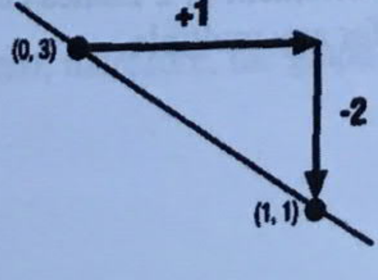
Note: The health bar situation corresponds to the section of this graph located in quadrant I ($x > 0$ and $y > 0$).

4.

a) $y = -2x + 3$

b) As x increases by one unit, y decreases by two. In general, y decreases by twice the x-increment.

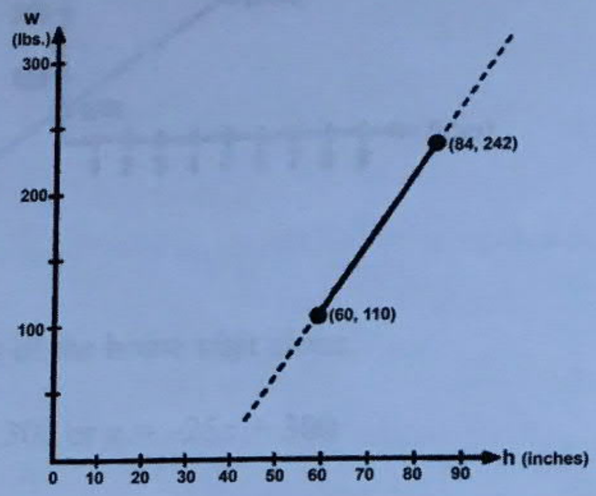
Ex.



III.

1.

a)



Note: Most adult males' heights lie between 60 inches (5 feet tall) and 84 inches (7 feet tall).

b)

Input (h)	Output (w)
60	110
62	121
64	132
66	143
68	154
70	165
72	176
74	187
76	198
78	209
80	220
82	231
84	242

c) 176 lbs. (6 feet = 72 inches)

d) 76 inches (or 6 feet and 4 inches)

e) To an increment of 1 inch (in height) corresponds to an increment of 5.5 pounds (in weight). Equivalently, to an increment of 2 inches corresponds to an increment of 2 × 5.5 = 11 pounds. And so on for any multiple.

Lesson 7
The Geometry of Linear Function Graphs

I.

1.
   a) A line.
   b) The line passes through the origin (0,0).
   c) The line is horizontal. (It neither “rises” nor “falls.”)
   d) The line equals the x-axis (the input axis or the horizontal axis) because its equation is $y = 0$.

2.
   a) The slope is positive; therefore, the line “rises.” The y-intercept is (0,–5); therefore, the line intersects the y-axis (the output axis or the vertical axis) five units below the origin.
   b) The slope is negative; therefore, the line “falls.” The y-intercept is (0,0); therefore, the line intersects the y-axis at the origin.
   c) The two lines have the same slope; therefore, they are parallel. The first intersects the y-axis at (0,3); the second, at (0,–1/3).
   d) The product of the two slopes is –1; therefore, the lines are perpendicular. The first intersects the y-axis at the origin; the second at (0,1).

II.

1.
   a) a is negative; $a = -1$
      b is positive; $b = 2$
   b) a is positive; $a = 2$
      b is zero
   c) a is zero
      b is negative; $b = -2.5 = -5/2$
   d) l_1: a is positive; $a = 1$
      b is positive; $b = 1$
      l_2: a is positive; $a = 1$
      b is negative; $b = -2$

   The lines are parallel because they have equal slopes ($a = 1$).

2. $f(x)$:
   The slope of its graph is zero; therefore, the graph is a horizontal line.
   The y-intercept is (0,7).

$g(x)$:
The slope of its graph is zero; therefore, the graph is a horizontal line.
The y-intercept is $(0,-3.5)$.

The graphs of these two functions are parallel to each other and to the x-axis.

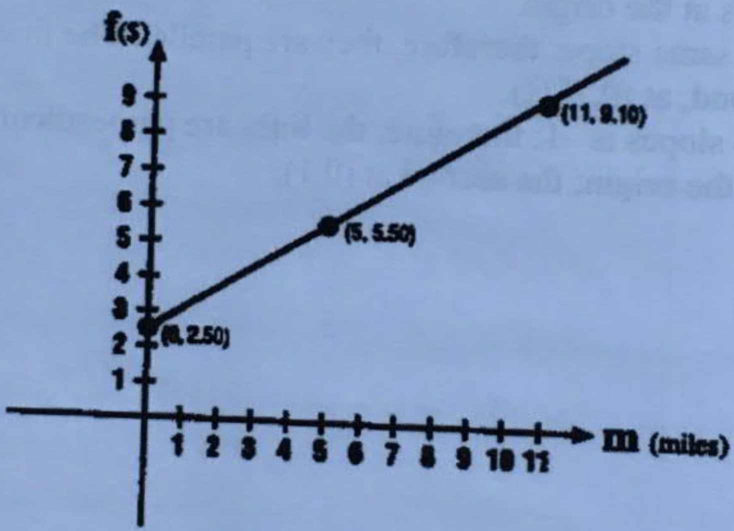
3.

a) An infinite number. Their common slope equals -0.2.
b) An infinite number. Their common slope equals $-1/-0.2 = +5$.
c) $y = -0.2x - 4$
d) $y = 5x + 5$

III.

1.

a) \$9.10 ($0.60 \times 11 + 2.50 = 9.1$)
b) 5 miles ($0.60 \times 5 + 2.50 = 5.50$)
c) $F = 0.60m + 2.50$ (or $y = 0.6x + 2.50$)
d)



Slope: $a = 0.60$: For each increment of 1 mile, there is an increase in fare of \$0.60.
y-intercept: $(0,b) = (0,2.50)$: The charge for using the taxi at the outset (before traveling the first mile, i.e., $m = 0$) is \$2.50.

Lesson 8
Words, Equations, Numbers, and Graphs

I.

1.

a) The verbal representation: The description (in English words) of a problem or situation (or what I often call a problem situation).

b) The symbolic representation: The algebraic or symbolic equation that summarizes the problem situation. [In the case of a linear function, the general form of the equation is $f(x) = ax + b$ or $y = ax + b$, which are equivalent.]

c) The numerical representation: The table of input/output values. This is a two-column table of numbers: the first column contains selected input values; the second contains the corresponding output values.

d) The graphical representation: The set of points (ordered pairs of numbers) plotted in the xy-plane. (In the case of a linear function, the points form a line.)

2. $y = 7x$. Let x represent the number of hours you babysit and y, your total earnings after x hours. The linear coefficient 7 represents your hourly rate in dollars. Example: If you babysat for 11 hours in one weekend, you would earn $y = 7 \times 11 = \$77.00$.

3.

a) See videotape.
b) \$33.00
c) 30 min. There are at least three ways to find this: 1) TRACE the graph of $y = 1.5x + 3$; 2) Use the TABLE feature; 3) Solve the equation $48 = 1.5x + 3$. (See Lesson 9.)

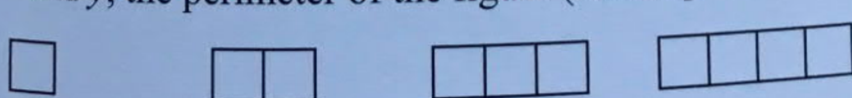
4.

a) (0,20)
b) 0
c) $y = -10$
d) $y = 0$ (This is the horizontal/input/x-axis.)
e) $f(x) = b$ or $y = b$, where b is a constant.

II.

1.

a) $a = 2$ and $b = 2$, so $y = 2x + 2$ or $f(x) = 2x + 2$. Situation: Let x be the figure number and y, the perimeter of the figure (each square has side-length 1):



x:	1	2	3	4	…
y:	4	6	8	10	…

b) (0,2). Not this particular situation.

2.

a) Use TBLSET and TABLE.

- Compute the slope: $a = \dfrac{y_2 - y_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are any two ordered pairs (or points).
- Find b in the output (or y) column. It corresponds to the input zero (or $x = 0$).
- Write $y = ax + b$ or $f(x) = ax + b$.

III.

1.

b) $b = 32$ (found at y-intercept)

c) $a = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{68 - 32}{20 - 0} = \dfrac{36}{20} = 1.8$

d) $f(x) = 1.8x + 32$ or $y = 1.8x + 32$

e) If x = degrees Celsius (or Centigrade) and y = degrees Fahrenheit, this equation is the conversion formula form C → F: F = 1.8C + 32. Ex.: 0 °C = 32 °F.

Lesson 9
Problem Solving with Linear Equations

I.

1.
   a) x = figure number; y = # of tiles in frame
   b) 80
   c) Figure #39 (Solve: $4x + 4 = 160$)

2.
   a) x = time (in hours); y = volume (in gallons)
   b) 31 hours and 15 minutes (Solve: $-240x + 15{,}000 = 7{,}500$)

3. The x-value (or input value) that corresponds to the y-value (or output value) of zero. Yes. The zero (or root) of $f(x) = 9x$ is 0 because $f(0) = 9 \times 0 = 0$. [**Note**: The *zero* of a function is not always zero. Ex.: The zero of $f(x) = 3x - 3$ is 1 because $f(1) = 0$.]

4. If $a \neq 0$, $f(x) = ax + b$ (or $y = ax + b$) has exactly <u>one</u> zero. If $a = 0$, $b \neq 0$, we have a horizontal line passing through $(0,b)$, and therefore, $f(x)$ has no zeros.

5. At the x-intercept: The zero of the linear function is the x-coordinate of this point.

II.

1.
   a) $y = 3x + 2$ [or $f(x) = 3x + 2$]
   b) 27 (Solve: $3x + 2 = 83$)

2.
   a) $y = 4.5x + 5$ [or $f(x) = 4.5x + 5$]
   b) 20 hours (Solve: $4.5x + 5 = 95$)

3.
   a) 11/2
   b) 2
   c) 2/3

4. 7/5

III.

1.

a) $y = 5.25x - 20$ [or $f(x) = 5.25x - 20$]

b) 51.43 hours (or 51 hours and 26 minutes), but since you are paid by the hour, you must work 52 hours.

c) <u>Functional Exploration</u>: i) Trace the graph of $y = 5.25x - 20$ and locate $y = 250$; the corresponding x is the solution. ii) Use the TBLSET and TABLE. Locate 250 in the y-column, then find the corresponding x.

d) <u>Symbolic Manipulation</u>: Solve $5.25x - 20 = 250$. $5.25x = 270$ (by adding 20 to both sides). $x = 51.43$ (by dividing both sides by 5.25)

Lesson 10
Modeling Real-World Data with Linear Functions

I.

1.

a) $y = (2/3)x + 1.43$ (or $y = .67x + 1.43$) [Explanation: $\frac{4.1-2.1}{4-1} = \frac{2}{3} \approx .67$ = slope; thus, $y = .67x + b$. To find b, plug in the coordinates of one of the points, such as (1,2.1), and solve for b.]

b) .7; .7; .6; .5

c) The average of these is .625; therefore, $y = .625x + b$. To find b, plug in the coordinates of one of these points, such as (4,4.1), and solve for b. Using (4,4.1), b = 1.5. Therefore, $y = .625x + 1.5$ is a line of good fit.

2.

a) Answers will vary depending on the selection of points.

b) Same as 2.a)

c) Same as 2.a)

II.

1. To each increment of one drop corresponds an increment of .63 cm in the average diameter.

2. To each increment of one mile (driven by the used car) corresponds a decrement (or negative increment) of –.25 dollars. In other words, each extra mile causes a drop in price of $0.25.

3. Measurements will vary depending on objects selected and time of day.

III.

1.

a) The equation found is of the form $y = ax$, where a is a constant.

c) Equation will depend on the data entered.

d) (i) Trace line and search for the x-coordinate of 180 (180 inches = 15 feet); the answer (in inches) will be the corresponding y-coordinate.
(ii) Use TBLSET and TABLE. Locate 180 in the x-column; the answer (in inches) will be the corresponding value in the y-column.

Lesson 11
Linear Functions and Geometry

I.

1.

a) $(23.4 - 13)/(7.4 - 4.1) = 3.1515\ldots$
$(29.4 - 23.4)/(9.25 - 7.5) = 3.429$ approximately
$(37.2 - 29.4)/(11.75 - 9.25) = 3.12$
$(47.5 - 37.2)/(15.2 - 11.75) = 2.986$ approximately

b) Average is 3.1716. About 3/100 greater than the actual value of π: quite accurate!

c) $C = \pi D$ or $y = \pi x$ is of the form $y = ax + b$, where $a = \pi$ and $b = 0$.

2.

a) π is the ratio of the circumference to the diameter of any circle.

b) π times (a little more than 3 times)

c) 2π times (about 6.3 times)

3.

a) $A = 180n - 360$

b) Let $A = y$ and $n = x$; we have $y = 180x - 360$, which is of the form $y = ax + b$, where $a = 180$ and $b = -360$.

c) $A = 180(8) - 360 = 1080$ degrees

d) 15 sides [Because $180(15) - 360 = 2340$.]

II.

1.

a) $P = 4S$

b) Yes. Let $P = y$ and $S = x$; we have $y = 4x$, which is of the form $y = ax + b$, where $a = 4$ and $b = 0$.

c) $A = S^2$

d) No. Let $A = y$ and $S = x$; we have $y = x^2$ which is <u>not</u> of the form $y = ax + b$. (We will see that this is called a *quadratic function*.)

2. Answers will vary.

3.

a) $l = w + 20$. Yes. This equation is of the form $y = ax + b$, where $a = 1$ and $b = 20$.

b) $P = 2l + 2w = 2(w + 20) + 2w = 2w + 40 + 2w = 4w + 40$. Yes. This equation is of the form $y = ax + b$, where $a = 4$ and $b = 40$.

c) 25

d) 80
e) $w = 1,\ l = 21$

III.

1.

a) Answers will vary.
b) $c = \pi d$
c) $h = 3d$
d) The circumference c is greater than the height h because $\pi > 3$!

Lesson 12

Quadratic Functions—Introductory Explorations I

I.

1. 625 (25^2)
2. 240 $(15^2 + 15)$
3. 181 $[2(10^2) - 2(10) + 1]$
4.
   a) x^2
   b) x^2
   c) $2x^2$

5.

a) 0
b) x
c) $-2x$

6.

a) 0
b) 0
c) 1

7. $f(x) = ax^2 + bx + c$, where a, b, c are constants. ax^2 is the quadratic term, bx is the linear term, and c is the constant term.

II.

1.

a) 4, 9, 16, …
b)

n	t
1	1
2	4
3	9
4	16
5	25

2.

a) ***Note:** The values under D_1 are between two consecutive values under T.

Example: $D_1 = 3$ is between the T values of 1 and 4; $D_1 = 5$ is between the T values of 4 and 9, and so on. Further, the numbers to the right represent the differences between the two adjacent numbers on the left (e.g., T:4 and 9; D_1:5).

n	T	D_1
1	1	
2	4	3
3	9	5
4	16	7
5	25	9

The first differences are not constant; therefore, the relationship (or function) is not linear.

b) **See *Note above.**

D_1	D_2
3	
5	2
7	2
9	2

The second differences are constant. We suspect that we have a second-degree polynomial function, meaning, a quadratic function.

3.

a) $T = n^2$ (or $y = x^2$). It is the same mathematical function we encountered in exploration 1. However, this time, the figures are equilateral triangles rather than squares.

b) $7^2 = 49$

III.

1.

a) 12, 24, 40

b)

x	y
0	0
1	4
2	12
3	24
4	40

c) The graph is not a straight line. The graph resembles the graphs encountered in the explorations of Lesson 12.

d) **See *Note for the answer to Problem 2.**

x	y	D_1
0	0	
1	4	4
2	12	8
3	24	12
4	40	16

The first differences represent the additional toothpicks required when going from one figure to the next. We notice more specifically that the number of additional toothpicks is four times the number of toothpicks on one side of the new figure.

e) **See *Note for the answer to Problem 2.**

D_1	D_2
4	
8	4
12	4
16	4

The second differences being constant, we suspect a second-degree polynomial function, i.e., a quadratic function.

f) $f(x) = 2x^2 + 2x + 0$

Lesson 13
Quadratic Functions—Introductory Explorations II

I.

1.

a) We can arrange the tiles (or dots, or beans…) in the shape of consecutive triangles. Example:

1 3 6 10

b) $\frac{n(n+1)}{2} = \frac{n^2+n}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$

c) Quadratic term: $\frac{1}{2}n^2$. Linear term: $\frac{1}{2}n$. Constant term: 0.

d) $a = 1/2, b = 1/2, c = 0$

e) 105

2.

a) $a = -1, b = 12, c = 0$

b) QT: $-x^2$. LT: $12x$. CT: 0.

c) (6,36). For the length of 6 yards, we have the maximum possible area of 36 square yards. [**Note**: Since the width is also 6 in this special case, we have a square of side-length 6 and area 36.]

d) (0,0) and (12,0). The lengths of 0 and 12 both correspond to an area of zero.

II.

1.

a) 1, 9, 36, 100, 225

b) They are perfect squares.

c) $1^2 = 1;\ 3^2 = 9;\ 6^2 = 36;\ 10^2 = 100;\ 15^2 = 225$

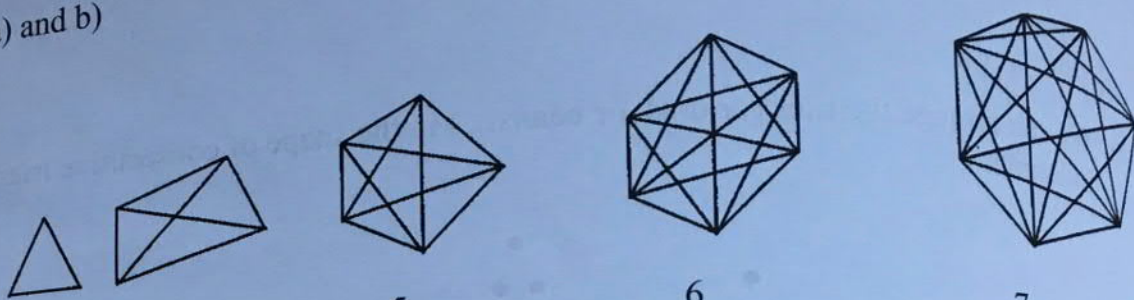
(**Note**: 1, 3, 6, 10, and 15 are the first five triangular numbers.)

d) <u>English</u>: The sum of the first n consecutive cubes equals the square of the nth triangular number. (Example: 225 is the sum of the first five cubes, and it is also the square of the fifth triangular number, 15.)

Algebra: Let $T, T_2, T_3, T_4 \ldots T_n$ denote the triangular numbers. $1^3 + 2^3 + 3^3 + \dots + n^3 = T_n^2$

2.

a) and b)



x: 3 4 5 6 7

Number of Sides (x)

y: 0 2 5 9 14

Number of Diagonals (y)

c)

Number of Sides (x)	Number of Diagonals (y)
3	0
4	2
5	5
6	9
7	14

d) Yes. Because the second differences are constant. (**Note**: The first differences are 2, 3, 4, 5, etc., and the second differences all equal 1.)

e) $\frac{x(x-3)}{2} = \frac{x^2-3x}{2} = \frac{1}{2}x^2 - \frac{3}{2}x$ (All three forms are equivalent.)

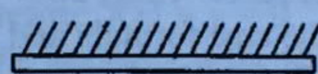
Explanation: x is the number of sides. We notice that a polygon with x sides also has x vertices. From each vertex emanate $(x-3)$ diagonals, because we subtract the vertex in question and the two adjacent vertices. If we count the diagonals from vertex 1 to vertex x, we have $x(x-3)$. But by doing so, we've counted each diagonal twice. Therefore, we must divide $x(x-3)$ by 2, hence the formula for the total number of diagonals in an x-gon.

III.

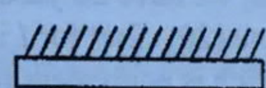
1.
a) We present the data on w, l, and A in the following table:

w	l	A
0	20	0
1	18	18
2	16	32
3	14	42
4	12	48
5	10	50
6	8	48
7	6	42
8	4	32
9	2	18
10	0	0

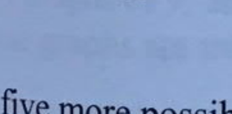
$w = 0, l = 20$
$A = 0$



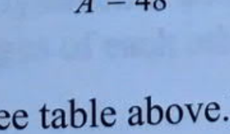
$w = 1, l = 18$
$A = 18$



$w = 2, l = 16$
$A = 32$



$w = 3, l = 14$
$A = 42$



$w = 4, l = 12$
$A = 48$

$w = 5, l = 10$
$A = 50$

There are five more possibilities (see table above.)

(**Note**: In each of these 11 cases, the total amount of fence is $2w + l = 20$. Check this fact to be convinced.)

b)

Width (w)	Area (A)
0	0
1	18
2	32
3	42
4	48
5	50
6	48
7	42
8	32
9	18
10	0

When you plot these points, you will find a graph similar in shape to that of exploration 2 in the videotape. (Let $w = x$ and $A = y$.)

c) $A = -2w^2 + 20w$ (or $y = -2x^2 + 20x$) [Explanation: $A = l{\times}w$. In this particular problem, we have 20 yards of fence. Thus, $2w + l = 20$ and $l = 20 - 2w$. We now substitute $(20 - 2w)$ for l in the area formula and obtain: $A = (20 - 2w)w = 20w - 2w^2$, which is equivalent to $-2w^2 + 20w$.]

d) Yes.

e) (5,50). When the width is 5 yards (and the length 10), the maximum possible area occurs and equals 50 square yards.

f) (0,0) and (10,0). When the width is either 0 or 10, we have a corresponding area of zero.

Lesson 14
The Geometry of Quadratic Function Graphs

I.

1.

a)

or

A parabola

b) It opens upward. Therefore, it has a lowest point.
c) It opens downward. Therefore, it has a highest point.
d) The y-intercept is on the y-axis. By definition, any point on this axis has an x-coordinate of zero. <u>Proof</u>: $f(0) = a0^2 + b0 + c = c$. $f(0) = c$ gives the point $(0,c)$.
e) Zero. An x-intercept is on the x-axis. By definition, any point on this axis has a y-coordinate of zero. Therefore, $f(x) = y = 0$.
f) Let $a > 0$: The vertex of the parabola is the lowest point on the graph. The y-coordinate of the vertex is the "absolute minimum"' value of the function. Let $a<0$: The vertex of the parabola is the highest point on the graph. The y-coordinate of the vertex is the "absolute maximum" value of the function.

2.

a) The graph of y_2 is the same graph as that of y_1, only shifted to the right five units. <u>Explanation</u>: $x^2 - 10x + 25 = (x-5)^2$. $(x-5)^2$ is obtained from x^2 by substituting $(x-5)$ for x (i.e., by subtracting 5 from x).
b) The graph of y_3 is the same as that of y_1, only shifted to the left five units. <u>Explanation</u>: $x^2 + 10x + 25 = (x+5)^2$. $(x+5)^2$ is obtained from x^2 by substituting $(x+5)$ for x (i.e., by adding 5 to x).
c) The graphs of y_2 and y_3 are symmetric with respect to the y-axis. We also say that these graphs are mirror images of each other with respect to the y-axis.

II.

1. Possible answers:
a) --The parabola opens upward.
--The y-intercept is (0,0).
--The x-intercept is (0,0).

b) --The parabola opens downward.

--The parabola is quite "narrow" (because of the value of a).
--The y-intercept is (0,5).

c) --The parabola opens upward.
--The parabola is quite "wide" (because the value of a is very small).
--The y-intercept is (0,2).

d) --The parabola opens upward.
--The y-intercept is (0,–3).
--The x-intercepts are (0,0) and (1,0). [Explanation: Factoring $f(x)$ gives $f(x) = 3x(x - 1)$ or $y = 3x(x - 1)$. It is very obvious in this form that when $x = 0$ or $x = 1$, the product $3x(x - 1)$ becomes zero; i.e., y becomes zero, hence the x-intercepts.]

2.

a) $f(x)$ opens upward; $g(x)$ opens downward.
Since $g(x) = -f(x) = -(2x^2 - 3x + 7) = -2x^2 + 3x - 7$, the graphs of f and g are symmetric with respect to the x-axis.

b) The graphs of f and g are symmetric with respect to the y-axis since $f(x) = (x - 2)^2$ and $g(x) = (x + 2)^2$.
Their graphs are identical in shape to that of the squaring function (x^2), only shifted to the right and left, respectively, by two units.
The vertex of $f(x)$ is (2,0).
The vertex of $g(x)$ is (–2,0).
(0,4) is the common y-intercept to both graphs.

c) Both graphs open upward.
Both graphs pass through (0,0).
The graph of $f(x)$ is a very "wide" parabola because a is very small.
The graph of $g(x)$ is a very "narrow" parabola because a is very large.

d) The graphs of $f(x)$ and $g(x)$ are identical in shape to that of $h(x) = 3x^2$.
The graph of $f(x)$ is the graph of $h(x)$ shifted upward two units.
The graph of $g(x)$ is the graph of $h(x)$ shifted downward nine units.
The vertex of $f(x)$ is (0,2).
The vertex of $g(x)$ is (0,–9).

III.

1. Yes. $f(x) = -x^2$
[Explanation:
--If (0,0) belongs to the graph of $f(x)$, then $f(0) = 0$. This implies $c = 0$ (I).
--If (1,–1) belongs to the graph of $f(x)$, then $f(1) = a(1)^2 + b(1) + 0 = -1$. This implies $a + b = -1$ (II).
--If (–1,–1) belongs to the graph of $f(x)$, then $f(-1) = a(-1)^2 + b(-1) + 0 = -1$. This implies $a - b = -1$ (III).

--Combining equations (II) and (III), we find $a = -1$ and $b = 0$. Combining the last two findings with equation (I), we obtain: $f(x) = -x^2 + 0x + 0 = -x^2$!]

Lesson 15
Words, Equations, Numbers, and Graphs

I.

1.

a)

x
0
1
2
3
4
5

b)

x	$y = ax^2 + bx + c$
0	c
1	$a + b + c$
2	$4a + 2b + c$
3	$9a + 3b + c$
4	$16a + 4b + c$
5	$25a + 5b + c$

c)

D_1
$a + b$
$3a + b$
$5a + b$
$7a + b$
$9a + b$

Note:

-- $a + b$ is the difference $(a + b + c) - c$
-- $3a + b$ is the difference $(4a + 2b + c) - (a + b + c)$
-- $5a + b$ is the difference $(9a + 3b + c) - (4a + 2b + c)$
-- $7a + b$ is the difference $(16a + 4b + c) - (9a + 3b + c)$
-- $9a + b$ is the difference $(25a + 5b + c) - (16a + 4b + c)$

d)

D_2
$2a$
$2a$
$2a$
$2a$

Note: These $2a$'s are the differences.

$(3a + b) - (a + b)$
$(5a + b) - (3a + b)$
$(7a + b) - (5a + b)$
$(9a + b) - (7a + b)$

e) Putting all four columns together, we obtain:

***Note:** The values under D_1 and D_2 are between two consecutive values under $y = ax^2 + bx + c$ and D_1. Example: $D_1 = a + b$ is between the $y = ax^2 + bx + c$ values of c and $a + b + c$; $D_2 = 2a$ is between the D_1 values of $a + b$ and $3a + b$, and so on. Further, all equations to the right represent the differences between the two adjacent numbers on the left; i.e., differences between $16a + 4b + c$ and $25a + 5b + c$: $D_1 = 9a + b$ and differences between the D_1 values of $7a + b$ and $9a + b$: $D_2 = 2a$.

x	$y = ax^2 + bx + c$	D_1	D_2
0	c		
1	$a + b + c$	$a + b$	
2	$4a + 2b + c$	$3a + b$	$2a$
3	$9a + 3b + c$	$5a + b$	$2a$
4	$16a + 4b + c$	$7a + b$	$2a$
5	$25a + 5b + c$	$9a + b$	$2a$

Note: The first expression (per column) is the simplest of its column. That is not to say that we cannot solve for a, b, and c using the other expressions, for example, $a + b + c$, $3a + b$, and $2a$ (which lie directly under the simplest expressions).

II.

1.

a) 1, 6, 15, 28. The fifth hexagonal number has 17 more dots than the fourth, or a total of 45 dots.

b)

x	y
0	0
1	1
2	6
3	15
4	28
5	45

Note: If there are zero dots on each side, there is a total of zero dots (or the 0th hexagonal number is 0).

c) ***Notes referring to "differences" from previous problems apply here also. (See Note for Answers, Lesson 12, Problem 2, and Note for Lesson 15, at the end of Section I.)**

y	D_1
0	
1	1
6	5
15	9
28	13
45	17

d) ***Notes referring to "differences" from previous problems apply here also. (See Note for Answers, Lesson 12, Problem 2, and Note for Lesson 15, at the end of Section I.)**

y	D_1	D_2
0		
1	1	
6	5	4
15	9	4
28	13	4
45	17	4

Note: See the correspondence between the numbers in this table and the expressions in Table I.1.e.

e) $c = 0$,
$a + b = 1$,
$2a = 4$. This last equation implies $a = 2$.
--Substituting 2 for a into the second equation yields $b = -1$.
[Explanation: $2 + b = 1$; (which is "equivalent to") $b = 1 - 2$; "equivalent to" $b = -1$.]
--The algebraic expression for the generalized hexagonal number is of the form $H(x) = ax^2 + bx + c$, because the second differences are constant. Substituting the numerical values we found above for a, b, and c, we obtain $H(x) = 2x^2 - x + 0$ or $H(x) = 2x^2 - x$.

f) $H(10) = 2(10)^2 - 10 = 200 - 10 = 190$

III.

1.

b)

n	Sequence
0	2
1	4
2	12
3	26
4	46
5	72

c) ***Notes referring to "differences" from previous problems apply here also. (See Note for Answers, Lesson 12, Problem 2, and Note for Lesson 15, at the end of Section I.)**

n	Sequence	D_1
0	2	
1	4	2
2	12	8
3	26	14
4	46	20
5	72	26

d) ***Notes referring to "differences" from previous problems apply here also. (See Note for Answers, Lesson 12, Problem 2, and Note for Lesson 15, at the end of Section I.)**

D_1	D_2
2	
8	6
14	6
20	6
26	6

e) $c = 2$,
$a + b = 2$,
$2a = 6$. This last equation implies $a = 3$.
--Substituting 3 for a into the second equation yields $b = -1$.
[Explanation: $3 + b = 2$ (which is "equivalent to") $b = 2 - 3$; "equivalent to" $b = -1$.]
Therefore, the nth number in this sequence is given by $f(n) = 3n^2 - n + 2$.

g) The continuous graph that would join all these points is a parabola.

Lesson 16
Problem Solving with Quadratic Equations

I.

1.

a) 16 [TRACE the graph of $f(n)$ or use TABLE.]
b) $(3/2)n^2 + (3/2)n = 408$ if and only if $(3/2)n^2 + (3/2)n - 408 = 0$, where $a = 3/2$, $b = 3/2$, $c = -408$, and $n = x$.
c) 135 [Compute $f(9)$ or use TABLE or TRACE the graph.]

2. Answers vary depending on objects selected.

3. $x^2 - x - 1 = 0$. It yields the golden ratio.

4.

a) The value(s) of x such that $f(x) = 0$ (they are found at the x-intercepts).
b) 0, 1, or 2

5. $f(x)$: -5 and $3/2$ (or 1.5)
6. $g(x)$ has no roots; its graph lies above the x-axis; therefore, $g(x)$ is never zero. (Use TABLE, or graph and use TRACE or CALC.)

II.

1.

a) 1.86 and -7.53
b) At the x-intercepts; more specifically, the x-coordinates of the x-intercepts.

2.

a) 1 (There is only one zero/root.)
b) We say the graph is *tangent* to the x-axis.

3. Example: $f(x) = 5x^2 + 15$ (There is an infinite number of examples.)

4. Example: $f(x) = -10x^2 + x$ (There is an infinite number of examples.)

5.

a) $120 - 4x^2$

b) 2"×2" squares.

[<u>Explanation</u>: We want $120 - 4x^2$ to equal 104, that is, $120 - 4x^2 = 104$ or $16 - 4x^2 = 0$.

<u>Graphical solution</u>: Graph $f(x) = 16 - 4x^2$. Find the roots, i.e., $x = -2$ and $x = 2$. Discard -2 since x represents the side length of a square and is thus positive.

<u>Algebraic solution</u>: We notice that $16 - 4x^2$ is the difference of two squares and, therefore, can be factored: $(4 + 2x)(4 - 2x) = 0$. The product of two factors is zero if either factor is zero:

$4 + 2x = 0$ when $x = -2$ and

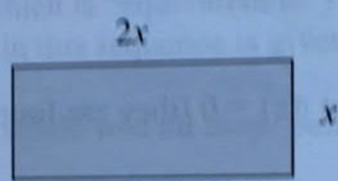
$4 - 2x = 0$ when $x = 2$.

Again, discard the negative solution. **Note:** You will see this algebraic approach in Algebra II.]

III.

1.

a) $2x$



b) $2x^2$ [The area of a rectangle equals length times width: $(2x)x = 2x^2$.]

c) 90 [We are told that $2x^2 = 450$ "or equivalently" $x^2 = 225$. 225 is the square of 15 and -15. We discard -15 since x represents the width of the rectangle. Therefore, the width is 15 feet, the length is 30 feet, and the perimeter (amount of fence) is $2(15 + 30) = 2(45) = 90$ feet.]

Lesson 17
Modeling Real-World Data with Quadratic Functions

I.

1.
a) See videotape.
b) An algebraic equation or a system of two or more equations.
c) It is usually expensive, time consuming, and complex (and often impossible) to study the real-world situation. Studying the mathematical model means studying a simplified and idealized simulation of the real-life situation.

2.
a) Input the data in two lists. (Press **STAT**, select 1: **Edit**.)
b) Turn **STAT PLOT** on, and select the appropriate lists that contain the data—the **Xlist** and the **Ylist**.
c) Press **ZOOM**, select 9: **ZoomStat** to plot the data.
d) Press **STAT** and the right arrow key to get the **EDIT CALC** menu. Select the appropriate regression.
e) The values of the numerical coefficients and the correlation coefficient are given by the graphing calculator.
f) Copy the regression equation into the $y=$ menu and graph.
g) Turn off **STAT PLOT** (select 4: **PlotsOff**).

3.
a) x: time in seconds since the object was released
y: distance in cm that the object has traveled
b) 3153.07 cm (Use **CALC**, select 1: Value and set $x = 2.5$.)

4.
a) x: ticket price in dollars
y: total annual revenue in thousands of dollars
b) \$157,052.50 (Use **CALC**, select 1: Value and set $x = 16.5$.)
c) \$13.16 (Use **CALC**, select 4: maximum.)

II.

1.
a) Press **STAT**, select 1: **Edit**.
b) Turn Plot 1 on [**STAT PLOT**] then press **ZOOM**. Select 9: **ZoomStat**.
c) $d = .0028s^2 + .1486s - .0857$ (The coefficients have been rounded off.)
d) Evaluating the equation in part c) for $s = 60$ yields $d = 18.91$.

III.

1.

c) $A = 2.7834r^2 + 2.8439r - 5.5353$ (The numerical coefficients have been rounded off.)

d) --Evaluating the equation in part c) for $r = 10$ yields $A = 301.24$.

--The actual value of the area of a circle whose radius is 10 is

$\pi r^2 = \pi(10)^2 = 100\pi = 314.16$.

This is not a very accurate model.

Lesson 18
Polynomial Explorations (Degree Greater than Two)

I.

1.
a) A cubic function. $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are constants.
b) A quartic function. $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d, and e are constants.

2.
a) Yes. Yes, because a volume is three-dimensional. $a = 4$, $b = -64$, $c = 256$, $d = 0$
b) 256 sq in $[f(x) = 4(4)^3 - 64(4)^2 + 256(4) = 256]$

3. ***Notes referring to "differences" from previous problems apply here also. (See Note for Answers, Lesson 12, Problem 2, and Note for Lesson 15, at the end of Section I.)**

x	$V = f(x)$	D_1	D_2	D_3
0	0			
1	196	196		
2	288	92	–104	
3	300	12	–80	24
4	256	–44	–56	24
5	180	–76	–32	24
6	96	–84	–8	24
7	28	–68	16	24
8	0	–28	40	24

a), b), and c) See table.

d) The volume function $f(x)$ is indeed a cubic function because the third differences are constant.

4.
a) 8: A cube has 8 vertices (or "corners").
12: A cube has 12 edges.
6: A cube has 6 faces.
b) $12(n - 2) = 12n - 24$ is of the form $ax + b$, where $a = 12$, $b = -24$, and $x = n$.
$6(n - 2)^2 = 6(n^2 - 4n + 4) = 6n^2 - 24n + 24$ is of the form $ax^2 + bx + c$, where $a = 6$, $b = -24$, $c = 24$, and $x = n$.
You saw in the videotape that $(n - 2)^3 = n^3 - 6n^2 + 12n - 8$; this is of the form $ax^3 + bx^2 + cx + d$, where $a = 1$, $b = -6$, $c = 12$, $d = -8$, and $x = n$.

II.

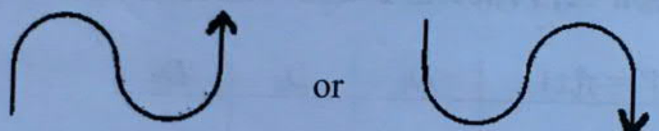
1.

a) Width: $0.5x$
b) Height: $0.4x$
c) $0.2x^3$ ($V = LWH = x(0.5x)(0.4x) = 0.2x^3$)
d) $L = 10$, $W = 5$, $H = 4$. Graph the volume function $V(x) = 0.2x^3$ from part c). TRACE the graph, looking for $y = 200$, or use TABLE and locate 200 in the y-column. Either way, you find $x = 10$. Thus, $L = 10$, $W = 0.5(10) = 5$, and $H = 0.4(10) = 4$.

III.

1.

a) and b) Use your graphing calculator to graph these and other cubic functions.
c) 1; 3; 2. A cubic function has either one, two, or three roots (or at least one and at most three).
d)



e) There is only one y-intercept. [To find its general expression, compute $f(0) = a(0)^3 + b(0)^2 + c(0) + d$, which yields $f(0) = d$. Therefore, the point $(0,d)$ is always the y-intercept of a cubic function graph.]

Lesson 19
Rational Functions—Introductory Explorations

I.

1.

a) $f(x) = \dfrac{n(x)}{d(x)}$, where $n(x)$ and $d(x)$ are polynomials.

b) $d(x)$ cannot equal zero. (When the denominator is zero, a fraction is undefined.)

c) The zeros of the function $f(x)$. [A fraction is zero if and only if its numerator is zero. Thus, $f(x) = 0$ if and only if $n(x) = 0$.]

2.

a) Example: $\dfrac{2x+100}{x^2+7x+10}$ (infinite examples)

b) Example: $\dfrac{-x^2+4}{6x^3+x}$ (infinite examples)

c) Example: $\dfrac{\frac{1}{2}x^3-2x^2}{\frac{3}{2}x+17}$ (infinite examples)

3.

a) A quadratic function. (degree 2)

b) A linear function. (degree 1)

c) x represented the length.
$f(x)$ represented the perimeter.

d) 20.85 [Evaluate $f(7)$, either by symbol manipulation or using the graph.]

e) $x = 1.82$ or $x = 13.18$. (TRACE the graph and locate the points for which $y = 30$, or use TABLE and locate the x-values for which $y = 30$.)

4.

a) The reciprocal function, because if you input a value for x, it outputs its reciprocal value.

b) It is very small.

c) It is very large.

d) It is very small.

e) It does not exist. (A fraction with a zero denominator is undefined.) There is a break in the graph at $x = 0$. (The vertical line $x = 0$ is called the *vertical asymptote* for this function.)

II.

1.

a) $x = 0$
b) $x = -5$
c) $x = 7$
d) $x = 1/2$

2.

a) $\frac{1}{x}$

b) $\frac{1}{x} + 4$

c) $x = 0.1$ or $1/10$ (Solve $\frac{1}{x} + 4 = 14$, either by graphical exploration or by symbol manipulation.)

III.

1.

a) \$80.00 [$30 + .25(200) = 80$]
b) $30 + .25x$ or $.25x + 30$
c) $\frac{.25x+30}{x}$ A rational function.
d) 500 miles. [Graph $f(x) = \frac{.25x+30}{x}$. Don't forget the parentheses around the numerator when you key in the fraction. Use TRACE or TABLE. In either case, look for $y = .31$; you find the corresponding $x = 500$.]

Lesson 20
The Geometry of Rational Function Graphs

I.

1.

a) No. The graph is interrupted at $x = 2$ (we say it is *discontinuous*). There is a vertical asymptote there whose equation is $x = 2$.

b) $(0,-1/2)$ or $(0,-0.5)$. To find the y-intercept of any graph, set $x = 0$ and evaluate $f(0)$.

c) No. To find the x-intercepts of any graph, set $y = 0$ and solve for x. But $\dfrac{1}{x-2} = 0$ yields no solution, because a fraction is zero if and only if its numerator is zero.

d) No.

e) No.

2.

a) One is the mirror image of the other with respect to the x-axis. In other words, if you reflect one about the x-axis, you obtain the other. [**Note**: This property holds for any function $f(x)$ and its "opposite," $-f(x)$.]

b) They have a common vertical asymptote, $x = 2$.

c) $(0,1/2)$ is the new y-intercept. We notice that the y-coordinates are opposites. That is not surprising since $y = f(x)$.

d) No x-intercepts for the same reason as I.1.c).

3.

a) The graph of $f(x) = 1/x$ shifted to the left 10 units.

b) The graph of $f(x) = 1/x$ shifted to the right 1/2 unit.

c) The graph of $f(x) = 1/x$ shifted to the left 5 units, then reflected about the x-axis.

II.

1.

a) Possible observations:

(i) $h(x)$ is always positive since its graph is entirely above the x-axis. (In quadrants I and II, $y > 0$.)

(ii) The vertical line $x = 0$, the vertical axis itself, is the vertical asymptote.

(iii) The graph is symmetric with respect to the y-axis; the right branch is a mirror image of the left. (<u>Explanation</u>: For any x value, $+x$ and $-x$ will yield the same y when plugged into $2/x^2$.)

b) <u>x-intercept</u>. Set $y = 0$: $2/x^2 = 0$ has no solution since the numerator cannot equal 0.
<u>y-intercept</u>. Set $x = 0$: But $f(0)$ is undefined, so $f(0)$ cannot be evaluated.

c) For $x \neq 0$, the numerator 2 and the denominator x^2 are always positive; therefore, the function $h(x) = 2/x^2$ is always positive.

d)

(i) All graphs, like that of $2/x^2$, live in quadrants I and II, i.e., above the x-axis.

(ii) All graphs, like that of $2/x^2$, have a vertical asymptote at $x = 0$.

(iii) While 2, 12, 22, and 32 increase by equal increments of 10, the graphs get progressively closer to each other as we go from 2 to 12, to 22, to 32…

2.

a) There are two values, +1 and –1, for which the denominator $x^2 - 1$ is zero. Therefore, the function $k(x)$ is undefined at both $x = -1$ and $x = +1$. Therefore, there are two asymptotes.

b) Three.

c) (0,–4) for the y-intercept; (–2,0) for the x-intercept.

Graphical Exploration: (i) TRACE the graph and locate the point where x is zero (y-intercept), then the point where y is zero (x-intercept). (ii) Use TABLE: Locate the value of y that corresponds to $x = 0$ (y-intercept), then the value of x that corresponds to $y = 0$ (x-intercept).

Symbol Manipulation: **Set x = 0:** $k(0)\dfrac{2\bullet 0+4}{0^2-1} = \dfrac{4}{-1} = -4$. Therefore, (0,–4) is the y-intercept. **Set y = 0:** $\dfrac{2x+4}{x^2-1} = 0$. [This is equivalent to $2x + 4 = 0$, or $2x = -4$, or $x = -2$. Therefore, (–2,0) is the x-intercept.]

III.

1.

a) Area: $A = 1/2(base) \times (height) = \dfrac{1}{2}yx$

b) $\dfrac{1}{2}yx = 5$

c) $\dfrac{1}{2}yx = 5$; multiplying both sides by 2 gives $yx = 10$; dividing both sides by x gives $y = 10/x$, which is y in terms of x.

d) Algebraically:

(i) $x = 5$ Evaluate: $y = 10/5 = 2$.

(ii) $y = 1.25$ Solve: $1.25 = 10/x$, which is equivalent to $1.25x = 10$ or $x = 10/1.25 = 8$.

Graphically:

(i) Locate the point on the graph of $y = 10/x$ where $x = 5$ and determine the corresponding y value: $y = 2$.

(ii) Locate the point on the graph of $y = 10/x$ where $y = 1.25$ and determine the corresponding x value: $x = 8$.

[**Note**: You can also use TABLE: Locate 5 in the x-column, determine y ($y = 2$); locate 1.25 in the y-column, determine x ($x = 8$).]

Lesson 21
Working with Rational Functions and Equations

I.

1.

a) --The graph lies in quadrants I and III.
--The y-axis ($x = 0$) is the vertical asymptote.
--The greater the a-value, the farther the graph from the axes.

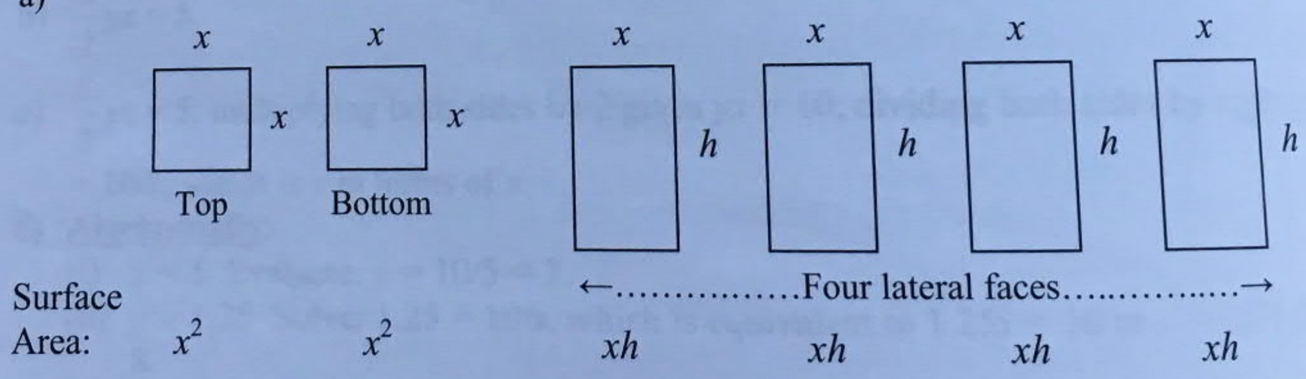
b) --The graph lies in quadrants II and IV.
--The y-axis ($x = 0$) is the vertical asymptote.
--The smaller the a-value, the farther the graph from the axes.

2.

a) The average speed, in mph, during the 250-mile trip.
b) The time, in hours, it took to make the 250-mile trip.
c) $S = 250/t$ or $y = 250/x$ (since $St = 250$)
d) $x = 4.55$ hours, or 4 hours and 33 minutes. Yes.
[Explanation: (i) Graph $y = 250/x$ with $x_{min} = 0$, $x_{max} = 15$, $y_{min} = 0$, $y_{max} = 120$. TRACE the graph and locate the point where $y = 55$. Zoom in and find $x \cong 4.55$. (ii) Use TblSet, with TblMin = 0, ΔTbl = .5. TABLE and locate the closest value to 55 in the y-column. Then refine the TblSet until you find the x-value that corresponds to $y = 55$, $x \cong 4.55$.]
e) $y = 93.75$ mph. Yes. [Explanation: (i) Evaluate y when $x = 2\,⅔$ hours. (ii) Graph $y = 250/x$ and use CALC, value and set $x = 2\,⅔$. (iii) Use TblSet and TABLE, searching for $x = 2.66$ in the x-column, then determine the corresponding y-value.]

3.

a)



b) Total surface area: $2x^2 + 4xh$
c) The volume is 250 and is expressed by *base* × *height*; thus, $x^2h = 250$, or equivalently, $h = 250/x^2$. Then we substitute $250/x^2$ for h in the surface area formula, which gives:

$2x^2 + 4x(250/x^2) = 2x^2 + 4(250x)/x^2 = 2x^2 + 1000/x = (2x^3 + 1000)/x$ (which is the function we graphed).

II.

1. a) Do not forget the parentheses when keying in the function into your graphing calculator: $(2x + 40)/(x^2 - 1)$.
   b) For x between -2 and -1 ($-2 < x < -1$), and for x greater than 1 ($x > 1$).
   c) For $x = -2$.
   d) For x less than -2 ($x < -2$), and for x between -1 and 1 ($-1 < x < 1$).

[Summarizing the results of b), c), and d)]

e) The graph of $k(x) = \dfrac{2x+4}{x^2-1}$ is above the x-axis when the quantity $\dfrac{2x+4}{x^2-1}$ is positive. It is below when this quantity is negative, and it intersects the x-axis when this quantity is zero. The combination of the signs of $(2x + 4)$ and $(x^2 - 1)$ gives the sign of the quotient. The following chart gives the analysis of these signs. (Note that $x^2 - 1 = (x + 1)(x - 1)$, so the denominator is broken down into two factors):

x	$-\infty$	-2		-1		0		1	$+\infty$
2x+4	-	0	+		+		+		+
x+1	-		-	0	+		+		+
x-1	-		-		-		-	0	+
sign of $(2x+4)/(x^2-1)$	- below	0	+ above		- below		- below		+ above
		↑ Intersects the x-axis		↑ Undefined				↑ Undefined	

2.

a) $2x$

b) $2 \cdot \dfrac{1}{x} = \dfrac{2}{x}$

c) $2x + \dfrac{2}{x} = \dfrac{2x}{1} + \dfrac{2}{x} = \dfrac{2x^2+2}{x}$

d) $\dfrac{2x^2+2}{x} = 5$

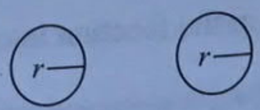
e) Graph $y = \dfrac{2x^2+2}{x}$. Solutions: $x = .5$ and $x = 2$ (Because we find that for $x = .5, y = 5$ and for $x = 2, y = 5$.)

f) Solutions: $x = .5$ and $x = 2$ (Because we find that $y = 5$ corresponds to both these values in the x-column.)

III.

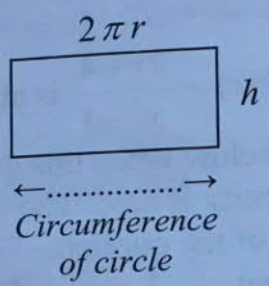
1.

a)



	Top	Bottom	} Top and Bottom Combined:
Surface Area:	πr^2	πr^2	} $\pi r^2 + \pi r^2 = 2\pi r^2$

b) Surface area: $2\pi r \bullet h = 2\pi rh$



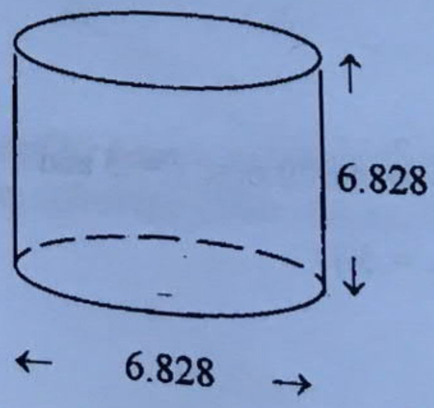
c) Total surface area: $S = 2\pi r^2 + 2\pi rh$

d) $S(r) = 2\pi r^2 + \dfrac{500}{r} = \dfrac{2\pi r^3 + 500}{r} \approx \dfrac{6.283\pi r^3 + 500}{r}$

[Explanation: Volume = Base × Height = $\pi r^2 h = 250$. Dividing both sides by πr^2, we obtain $h = 250/\pi r^2$. Substituting this expression for h in part c) yields:

$2\pi r^2 + 2\pi r\left(\dfrac{250}{\pi r^2}\right) = \dfrac{2\pi r^2}{1} + \dfrac{500}{r} = \dfrac{2\pi r^3 + 500}{r}$]

e) Graph $y = (6.283x^3 + 500)/x$ with WINDOW limits $x_{min} = 0$, $x_{max} = 7$, $y_{min} = 0$, $y_{max} = 350$. Do not forget to place parentheses around the numerator. Use CALC (#3 minimum) to calculate the minimum value in quadrant I. You find $x = 3.4139$. Substituting this value for r in the formula $h = \dfrac{250}{\pi r^2}$ yields $h = 6.8279$. But x is the radius; therefore, the diameter is $2x = 2(3.4139) = 6.8278$. While the diameter and the height seem to differ slightly, in fact, they are equal. Result: The dimensions of the can that will use the least amount of tin are: diameter = height $\cong$ 6.828.



Lesson 22
Exponential Functions—Introductory Explorations

I.

1.

a) Exploration 1: Square #1 had 2^0 kernels of wheat, square #2 had 2^1, square #3 had 2^2, and so on. Thus, 2^x represented the number of kernels on square #$(x + 1)$.

b) Exploration 2: We began with a sheet of paper. After 1 fold we had 2^1 layers of paper, after 2 folds we had 2^2, after 3 folds we had 2^3, and so on. Thus, 2^x represented the number of layers (or sheets) of paper after x folds.

2.

a) Graph $f(x) = 2^x$ with the following WINDOW limits: $x_{min} = -5$, $x_{max} = 5$, $y_{min} = -2$, $y_{max} = 15$.

b) Possible statements:
--The set of input values (x-values) is the set of all real numbers.
--The set of output values (y-values) is the set of all positive real numbers.
--The graph of $f(x) = 2^x$ therefore lies in quadrants I and II.
--The graph has no x-intercepts.
--The y-intercept is $(0,1)$.
--$f(x) = 2^x$ is an increasing function. (as x increases, y increases)

3.

a) 16
b) 512
c) No, because we have exponential growth. [For example, for $f(x) = 3x$, $f(5) = 15$ and $f(10) = 30$; in this case, we do have $f(10) = 2f(5)$.]

4.

a) About 56.5 miles high.

b)

After 9 folds
After 12 folds
After 14 folds
After 25 folds
Note: Use TBLSET and TABLE for part b).

II.

1.

a) The input variable (x) is the base, not the exponent. This is a <u>cubic</u> function.

b) This function can be written equivalently as $3x^{-1}$. Again, the input variable (x) is not the exponent. This is a <u>rational</u> function.

c) Here, both the exponent and base are constants. This is a <u>constant</u> function: $f(x) = 25$.

2.

a) If we change the base to 3, 4, or 5, the equation no longer holds.

b) It is a true statement.

c) $(1/2)^0 = 2^0$; $(1/2)^{-1} = 2^1$; $(1/2)^{-2} = 2^2$; $(1/2)^{-3} = 2^3$; etc. Thus, this equation is equivalent to $2^0 + 2^1 + 2^2 + ... + 2^n = 2^{n+1} - 1$.

3.

a)

--The set of input values (x-values) is the set of all real numbers.
--The set of output values (y-values) is the set of all <u>positive</u> real numbers.
--The graph lies in quadrants I and II.
--The graph has no x-intercepts.
--The y-intercept is (0,1).

b) The graph of $y = 2^x$ and $y = (1/2)^x$ are symmetric with respect to the output axis (the y-axis/the vertical axis). $y = 2^x$ is an increasing function and $y = (1/2)^x$ is a decreasing function. The same relationship holds between $y = 5^x$ and $y = (1/5)^x$.

III.

1.

a)

<u>Job offer 1</u>: \$150 less
<u>Job offer 2</u>: 2^{30} cents less, or \$10,737,418.24 less.

b) <u>Job offer 1</u>: \$4650 [\$150 × 31]
<u>Job offer 2</u>: \$21,474,836.47 [$(2^0 + 2^1 + 2^2 + ... + 2^{30}) = (2^{31} - 1)$ cents]

c) <u>Job offer 1</u>: Day 1–15: \$2250 [\$150 × 15]
Day 16–31: \$2400 [\$150 × 16]
<u>Job offer 2</u>: Day 1–15: \$327.67 [$(2^{15} - 1)$ cents]
Day 16–31: \$21,474,508.80

d) $f(x) = 150$ and $g(x) = 2^{x-1}/100$ would give the respective <u>daily</u> earnings in dollars: $f(x) = 150$: x, the input variable, takes on discrete values from 1 to 31, representing the 31

days of August. $f(x)$ or y, the output variable, constantly equals 150, representing the daily earnings in dollars.

$g(x) = 2^{x-1}/100$: x, the input variable, takes on discrete values from 1 through 31, representing the 31 days of August. $g(x)$ or y, the output variable, equals $2^{x-1}/100$, which represents the daily earnings in dollars.

Lesson 23
The Geometry of Exponential Function Graphs

I.

1.

a) x^3
b) y (or y^1)
c) 1 (or z^0)
d) w^{-3} [or $(1/w)^3$ or $1/w^3$]
e) v^6

2.

a) 2; 2
b) 4; 4
c) 64; 64
d) For any real number x, we have $f(x) = g(-x)$ [i.e., $2^x = (1/2)^{-x}$].

3.

a) --Their graphs lie entirely above the x-axis (i.e., the functions are positive).
--They all share a common y-intercept (0,1).
b) $f(x) = 1^x$ is a constant function; it neither increases nor decreases; therefore, it is a trivial case. No.
c) If b is negative, there are many x values, such as 1/2, 1/4, 1/6, 1/8…, for which b^x is undefined (or does not exist).

II.

1.

a) They are symmetric (or mirror images of each other).
b) (0,1)
c) It approaches the x-axis (horizontal axis) while never actually reaching it.
d) Same as d).

2.

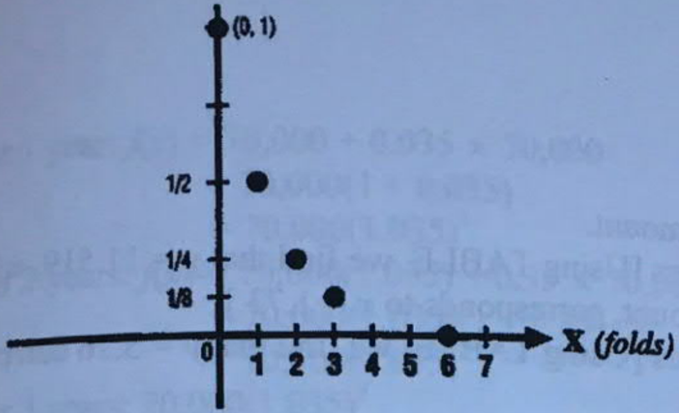
a) 8
b) 1024
c) $f(x) = 2^x$ (Another example of the doubling function!)

3.

a) 1/4; 1/8
b) 1/64

c) $f(x) = (1/2)^x$ or $1/2^x$

d)



Note: It is the discrete analogue of the continuous graph $f(x) = (1/2)^x$.

III.

1.

a) $.00389x + 2 = ax + b$, where $a = .00389$ and $b = 2$.

c) 2.46 billion; 5.04 billion [Use TRACE and ZOOM IN, or CALC, (1) Value, or TBLSET and TABLE. Locate the y-values corresponding to $x = 1900$ and $x = 1980$.]

d) Between 1999 and the year 2000. [Use TRACE and ZOOM IN, or TBLSET and TABLE. This time, locate the x-value corresponding to $y = 6$ billion.]

Lesson 24
Working with Exponential Functions and Equations

I.

1.

a) The initial (or original) amount.
b) Approximately 1,730 years [Using TABLE, we find that $y = 11.519$, which is about one-half of the initial amount, corresponds to $x = 1.73$.]
c) Approximately 3,460 years [Using TABLE, we find that $y = 5.76$ corresponds to $x = 3.46$.]

2.

a) 1905. 10 years.
b) Use your graphing calculator.
c) (11.32, 96.97). Around the year 2018, the men's and women's world records will both equal 96.97.
d) men's: 100.19; women's: 106.6
e) men's: 98.79; women's: 102.33

II.

1.

a) Yes. f: $A = 1$; $b = 2.5$. g: $A = 3$; $b = 2.5$. h: $A = 7.5$; $b = 2.5$.
b) Quadrants I and II. The portions that lie in quadrant I.
c) Exponential growth. As x increases, $f(x)$ or y increases (or, as the input increases, the output increases.)
d) f: (0,1). g: (0,3). h: (0,7.5). The initial amounts (or the starting values.)

2.

a) Yes. f: $A = 1$; $b = 0.4$. g: $A = 4.5$; $b = 0.4$. h: $A = 9$; $b = 0.4$.
b) Quadrants I and II. The portions that lie in quadrant I.
c) Exponential decay. As x increases, $f(x)$ or y decreases.
d) f: (0,1). g: (0,4.5). h: (0,9). The initial amounts.

3.

a) The input variable (or the independent variable) x is in the place of the exponent.
b) When $A = 1$, $Ab^x = b^x$.
c) A is the y-coordinate of the y-intercept $(0,A)$. Real-world situations usually begin at $x = 0$, $(y = A)$, and since we study the "growth" or "decay" of some existing quantity A, A must be positive at the start.
d) If $b > 1$, the graph "rises"; therefore, exponential growth.

If $0 < b < 1$, the graph "falls"; therefore, exponential decay.

III.

1.

a) After 1 year: $f(x) = 70{,}000 + 0.035 \times 70{,}000$
$= 70{,}000(1 + 0.035)$
$= 70{,}000(1.035)^1$

After 2 years: $f(x) = 70{,}000(1.035) + 0.35 \times 70{,}000(1.035)$
$= 70{,}000(1.035)(1 + 0.35)$
$= 70{,}000(1.035)^2$

After 3 years: $70{,}000(1.035)^3$

After x years: $70{,}000(1.035)^x$

b) Yes. $A = 70{,}000$. $b = 1.035$.

c) 92,117.[$f(8) = 92{,}117$]

d) In 2007 [On Jan. 1, 2007, $f(x) < 100{,}000$; on Jan. 1, 2008, $f(x) > 100{,}000$.]

e) In 2017 [On Jan. 1, 2017, $f(x) < 140{,}000$; on Jan. 1, 2018, $f(x) > 140{,}000$.]

Lesson 25
Systems of Linear Functions and Equations

I.

1.

a)

$x + y = 5$
$x - y = 3$

[**Note**: Any multiples of these equations are also acceptable since they are equivalent. For example, $-x + y = -3$ is obtained by multiplying the second equation by (-1) and is, therefore, equivalent to it.]

b)

$(1/2)x - y = -2/3$
$(3/2)x + y = 1/4$
or
$3x - 6y = -4$
$6x + 4y = 1$

c)

$1.7x - y = -10$
$10x + y = -1.7$

2.

a) Equation I. Equation II.
b)
$s + c = 35$
$3s + 4c = 122$
c) 18 stools, 17 chairs
d) Same as c).
e) Same as c). [Solve both equations for c. You obtain $c = -s + 35$ or $y = -x + 35$, and $c = (-3/4)s + 61/2$ or $y = -0.75x + 30.5$. Graph these equations with WINDOW limits: $x_{min} = y_{min} = 0$ and $x_{max} = y_{max} = 40$. The intersection point of the graphs is (18,17).]
f) $18 + 17 = 35$;
$3(18) + 4(17) = 122$. Both check.

3.

a) When there is no (simultaneous) solution. The lines are parallel.
b) When there is a unique solution. The lines intersect at one point, and its coordinates constitute the unique solution.

c) Where there are infinitely many solutions. The lines coincide, and the coordinates of all the points on the line constitute the solutions.

II.

1. $x = 37/23$;
   $y = -6/23$
2. No solution.
3. An infinite number of solutions.
4. $x = -11/14$
   $y = -11/24$

III.

1.

a) $c + b = 4$
b) $8c + 7.5b = 30.75$
c) 1.5 pounds of Colombian coffee and 2.5 pounds of Brazilian coffee.
d) $1.5 + 2.5 = 4$
$8(1.5) + 7.5(2.5) = 12 + 18.75 = 30.75$. Both check.

Lesson 26
Using Matrices to Solve Linear Systems

I.

1.

a) $\begin{bmatrix} 1 & 1 \\ 3.75 & 1.25 \end{bmatrix}$

b) $\begin{bmatrix} x \\ y \end{bmatrix}$

c) $\begin{bmatrix} 120 \\ 307.50 \end{bmatrix}$

d) $AX = B$

2.

a) $63 + 57 = 120$
$3.75(63) + 1.25(57) = 236.25 + 71.25 = 307.50$ Both check.

b) $y_1 = -x + 120$
$y_2 = -3x + 246$
The graphs of these equations intersect at (63,57).

3.

a) 2×2; 2×1; 2×1
b) 3×3; 3×1; 3×1

II.

1. A: 2×3
B: 3×2
C: 1×2

2.

a) $A + B = \begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix}$ $\qquad A \times B = AB = \begin{bmatrix} 0 & 8 \\ 6 & 14 \end{bmatrix}$

b) $B + A = \begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix}$ $\qquad B \times A = BA = \begin{bmatrix} 0 & 4 \\ 12 & 14 \end{bmatrix}$

c) Matrix addition is commutative.
Matrix multiplication is not commutative.

3.

a) $A^{-1}=\begin{bmatrix} 5 & -2 \\ -1 & .5 \end{bmatrix}$

b) $AA^{-1}=A^{-1}A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c) $AA^{-1}=A^{-1}A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

d) $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. For any matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $AI = IA = A$.

<u>Proof</u>: $AI=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\bullet\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=A$

$$IA=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\bullet\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=A$$

e) $I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

III.

1.
$a + s = 500$
$7a + 4s = 2900$

$A=\begin{bmatrix} 1 & 1 \\ 7 & 4 \end{bmatrix}$ $X=\begin{bmatrix} a \\ s \end{bmatrix}$ $B=\begin{bmatrix} 500 \\ 2{,}900 \end{bmatrix}$

$AX = B$, or equivalently, $X = A^{-1}B = \begin{bmatrix} 300 \\ 200 \end{bmatrix}$. Therefore, 300 adult tickets and 200 student tickets were sold.

2. $n + d + q = 95$
$.05n + .1d + .25q = 11.25$
$2n - d = 0$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ .05 & .1 & .25 \\ 2 & -1 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} n \\ d \\ q \end{bmatrix} \qquad B = \begin{bmatrix} 95 \\ 11.25 \\ 0 \end{bmatrix}$$

$AX = B$ is equivalent to $X = A^{-1}B = \begin{bmatrix} 25 \\ 50 \\ 20 \end{bmatrix}$.

Therefore, 25 nickels, 50 dimes, and 20 quarters are in this collection.

Lesson 27
Systems of Functions and Equations

I.

1. There are three cases:
   Case 1: The line and the parabola do not intersect; therefore, there is no solution.
   Case 2: The line is tangent to the parabola at one point; therefore, there is a unique solution.
   Case 3: The line intersects the parabola at two distinct points; therefore, there are two solutions.

2. There are three cases:
   Case 1: The line and the hyperbola do not intersect; therefore, there is no solution.
   Case 2: The line is tangent to the parabola at one point; therefore, there is a unique solution.
   Case 3: The line intersects the parabola at two distinct points; therefore, there are two solutions.

3. We are looking for the side length of a square, and a length cannot be negative.

4.

a)

$l = 2w + 4$ $l = 2w + 4$ $y = 2x + 4$

$lw = 70$ or $l = 70/w$ or $y = 70/x$

b) The width is 5 and the length is 14.

c) $14 = 2(5) + 4$ and
$14(5) = 70$. Both check.

5.

a) $(-.766, +.587)$; $(2,4)$; $(4,16)$

b) The two graphs intersect at these three distinct points.

c)

$(-.766)^2 \approx .587$ and $2^{-.766} \approx .587$

$2^2 = 4$ and $2^2 = 4$

$4^2 = 16$ and $2^4 = 16$

II.

1.

a)

$x + y = 30 \quad y = -x + 30$

$xy = 221 \quad \text{or} \quad y = 221/x$

b) and c) Solutions: (13,17) and (17,13). Their positive difference is 4.

2.

a) (3,23)
<u>Functional exploration</u>: (i) Graph functions with WINDOW limits: $x_{min} = y_{min} = -5$, $x_{max} = 5$, $y_{max} = 25$. CALC intersect yields (3,23). (ii) TBLSET: TblMin = 0, ΔTbl = 1. TABLE: At $x = 3$, $y_1 = y_2 = 23$.
<u>Symbolic manipulation</u>: $2x^2 + 5 = 2x^2 + 3x - 4$ is equivalent to $5 = 3x - 4$ or $3x = 9$ or $x = 3$. Evaluating either function for $x = 3$ gives $y = 23$.

b) (–2.56, –1.56) and (1.56, 2.56)
<u>Functional exploration</u>: (i) Graph $y = -x^2 + 5$ and $y = x + 1$ in the standard window (ZOOM 6: ZStandard). Use CALC intersect twice. (ii) Using TBLSET and TABLE would be more time-consuming.
<u>Symbolic manipulation</u>: Algebra I does not provide the algebraic methods to solve $-x^2 + 5 = x + 1$ or $-x^2 - x + 4 = 0$.

c) No solution.
<u>Functional exploration</u>: Graph $y = 5/x$ and $y = -x$ in the standard window (ZOOM 6) and notice that the graphs do not intersect.
<u>Symbolic manipulation</u>: $5/x = -x$ is equivalent to $5 = -x^2$ or $x^2 = -5$. We see that this is impossible to solve since $x^2 \geq 0$.

III.

1.

a) Quadrants I and III.
b) Quadrants I and II.
d) (2.478, 15.217) and (3,27) Graph f and g in Quadrant I with WINDOW limits: $x_{min} = y_{min} = 0$, $x_{max} = 4$, $y_{max} = 50$. Use CALC intersect twice.)

Lesson 28
Systems of Inequalities

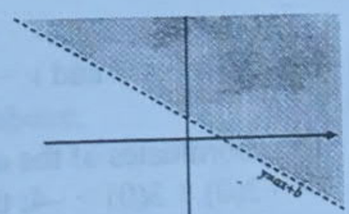
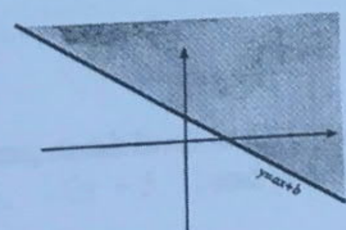
I.

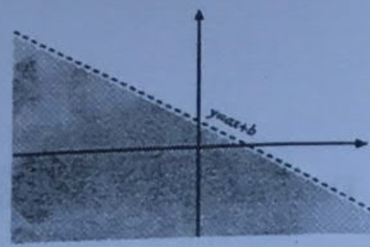
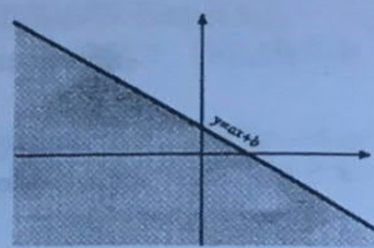
1.

a) There are four possibilities:

Inequality:	Number Line:	Interval Notation:
i. $x < a$		$(-\infty, a)$
ii. $x \le a$		$(-\infty, a)$
iii. $x > a$		$(a, +\infty)$
iv. $x \ge a$		$(a, +\infty)$

b) There are four possibilities:

Inequality:	Explanation:	*xy*-Plane:
i. $y > ax + b$	The region of points lying above the boundary line.	
ii. $y \ge ax + b$	The region of points lying on or above the boundary line.	

Inequality:	Explanation:	xy-Plane:
iii. $y < ax + b$	The region of points lying below the boundary line.	
iv. $y \leq ax + b$	The region of points lying on or below the boundary line.	

c) A variety of possibilities, from the empty set to the set of points lying in a half-plane. The solution set is the intersection of the solution sets of the two respective inequalities (See b).

2. The solution set of $-3x + 4 < 2$ does not include $x = 2/3$.

3. The solution set of $2x - y \leq 3$ includes all ordered pairs (x,y) that lie <u>on</u> the line $y = 2x - 3$.

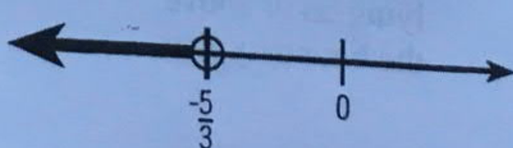
4.

a) $y = (2/3)x - 4/3$ and $y = -x + 3$
b) Four.
c) The coordinates of the origin verify both inequalities:
$-2(0) + 3(0) > -4$; therefore, $0 > -4$ True.
$0 + 0 < 3$; therefore $0 < 3$ True.

II.

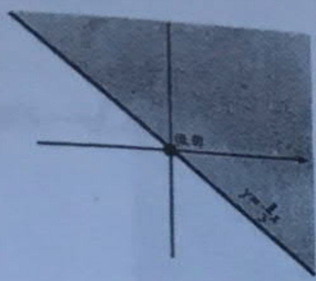
1.

a) $x < -5/3$

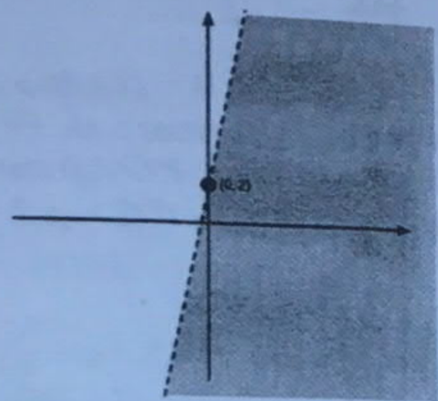


b) $x \leq 1.\overline{891}$

c) $y \geq (-8/3)x$

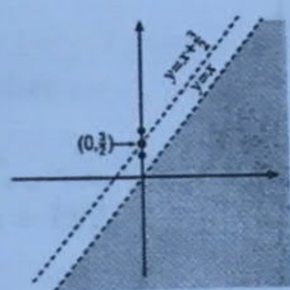


d) $y < 5x + 2$

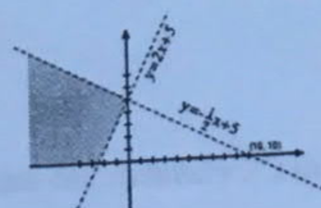


2.

a) Graph $y = x$ and $y = x + 3/2$. Solve each inequality separately, then graph the intersection of the two solution sets.

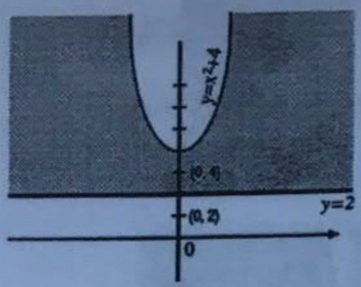


b) Graph $y = (-1/2)x + 5$ and $y = 2x + 5$. Proceed as above.

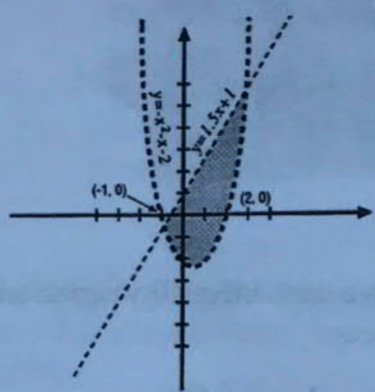


Note: You may use the SHADE (lower func, upper func, resolution) feature on your graphing calculator. In this case, type SHADE $(2x + 5, -1/2x + 5, 1)$ and you obtain the above region of solutions.

c) SHADE $(2, x^2 + 4, 1)$ gives:



d) SHADE $(x^2 - x - 2, 1.5x + 1, 1)$ gives:



III.

1.

a) $y = -x + 1; y \leq -x + 1$
b) $y = x - 1; y \geq x - 1$
c) $y = -x - 1; y \geq -x - 1$
d) $y = x + 1; y \leq x + 1$
e)
$y \leq -x + 1$
$y \geq x - 1$
$y \geq -x - 1$
$y \leq x + 1$

Lesson 29
Iterating Functions—Looking at Functions Recursively

I.

1. Answers will vary.
2.
   a) $f(n) = 2f(n-1) + 1$, with $f(1) = 1$
   b) $x_5 = 9.6875$; $x_{10} \approx 9.99$. As n increases, x_n approaches 10. (Enter 5 on your graphing calculator. Then iterate 0.5ANS+5 the desired number of times.)
   c) 39. [$x_1 = 2(3) - 1 = 5$, $x_2 = 2(5) - 2 = 8$, $x_3 = 2(8) - 3 = 13$, $x_4 = 2(13) - 4 = 22$, $x_5 = 2(22) - 5 = 39$.]

3.

a)

Using the recurrence relation: You must iterate five times starting with F_3 and stopping at F_7 ($F_1 = F_2 = 1$ are given).
Using the explicit formula: You evaluate the formula for $n = 7$.

b) $F_3 = 2$, $F_4 = 3$, $F_5 = 5$

4. Let T_n be the nth triangular number.

a)

Recursive definition: $T_n = T_{n-1} + n$, where $T_1 = 1$.
Explicit definition: $T_n = n(n+1)/2$

b)

$T_6 = 6(7)/2 = 21$
$T_1 = 1$; $T_2 = 1 + 2 = 3$; $T_3 = 3 + 3 = 6$; $T_4 = 6 + 4 = 10$; $T_5 = 10 + 5 = 15$; and finally, $T_6 = 15 + 6 = 21$

II.

1.

a) In the first, 1 is subtracted from the quantity 2^n; in the second, 1 is subtracted from the exponent of 2, which is n.

b)

n	$2^n - 1$	2^{n-1}
–2	–.75	.125
–1	–.5	.25
0	0	.5
1	1	1
2	3	2
3	7	4

For $x < 1$, $2^n - 1 < 2^{n-1}$
For $x > 1$, $2^n - 1 > 2^{n-1}$
For $x = 1$, $2^n - 1 = 2^{n-1}$

c) Yes.

2. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

a) $F_2/F_1 = 1$; $F_3/F_2 = 2$; $F_4/F_3 = 1.5$; $F_5/F_4 = 1.66$; $F_6/F_5 = 1.6$; $F_7/F_6 = 1.625$; $F_8/F_7 = 1.615$; $F_9/F_8 = 1.619$; $F_{10}/F_9 = 1.617$; $F_{11}/F_{10} = 1.6182$; $F_{12}/F_{11} = 1.6179$

Observations:
(i) The ratios "converge" to the golden ratio $\varphi = 1.618034...$, which we saw in Lesson 16.
(ii) The values alternate between being a little greater than and a little less than the exact value of φ.

b) The greater the value of n, the closer the decimal value of F_n/F_{n-1} is to φ.

3.

a) Because x_n is defined in terms of the three preceding terms $x_{n-1}, x_{n-2}, x_{n-3}$ for $n \geq 3$.
b)
$x_3 = x_2 - x_1 + x_0 = 2 - 1 + 0 = 1$
$x_4 = x_3 - x_2 + x_1 = 1 - 2 + 1 = 0$
$x_5 = x_4 - x_3 + x_2 = 0 - 1 + 2 = 1$
$x_6 = x_5 - x_4 + x_3 = 1 - 0 + 1 = 2$
$x_7 = x_6 - x_5 + x_4 = 2 - 1 + 0 = 1$

0, 1, 2, 1, 0, 1, 2, 1, 0…

4.

a)

$l_1 = 0$ $\quad l_2 = l_1 + 1 = 0 + 1 = 1$ $\quad l_3 = l_2 = 2 = 1 + 2 = 3$ $\quad l_4 = l_3 + 3 = 3 + 3 = 6$ $\quad l_5 = l_4 + 4 = 6 + 4 = 10$

Note: The dotted-line segments at stage n are the new line segments added to l_{n-1} to obtain l_n.

b) $l_n = l_{n-1} + (n - 1)$ for $n \geq 2$, where $l_1 = 0$

c) $l_n = n(n - 1)/2$

III.

1.

a)

1 disk: 1 move
2 disks: 3 moves
3 disks: 7 moves
4 disks: 15 moves

b) $M_n = M_{n-1} + 1 + M_{n-1} = 2M_{n-1} + 1$ for $n \geq 2$, where $M_1 = 1$.

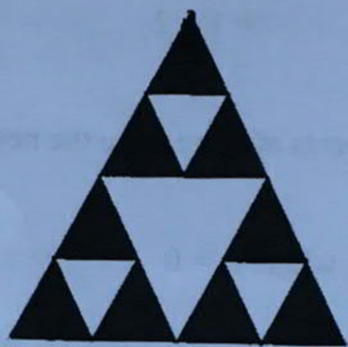
c) $M_n = 2^n - 1$

Lesson 30
Using Iteration as a Problem-Solving Tool

I.

1.

a) See videotape for details of the geometric approach and the chaos game.
b) Second iteration of the ST:



2.

a) $x_0 = 1; x_1 = 3; x_2 = 9; x_3 = 27; x_4 = 81$
b) $x_n = 3^n$ or $f(n) = 3^n$
c) $x_n = 3x_{n-1}$, with $x_0 = 1$ or $f(n) = 3f(n-1)$, with $f(0) = 1$

3.

a) $A_5 = \$7346.64$
$A_{10} = \$10{,}794.62$

b) $A_n = 5000(1.08)^n$ or
$f(n) = 5000(1.08)^n$

c) $A_n = (1.08)A_{n-1}$, with $A_0 = 5000$ or
$f(n) = (1.08)f(n-1)$, with $f(0) = 5000$

4.

a) \$421.32. The fact that there is still an outstanding balance of \$421.32 implies that a monthly payment of \$350.00 is not large enough.
b) $r = 1$ or $r = -1$ means that the regression line is a perfect fit. The negative sign signifies a decreasing function.
c) The monthly payment. The final balance after 4 years or 48 months.
d) (357.40 ,0) means that a monthly payment of \$357.40 will pay off the loan in 4 years, leaving a final outstanding balance of \$0.00.

(0, 20,347.34) means that a monthly payment of \$0.00 will yield a final outstanding balance of \$20,347.34 after 4 years.

e) \$17,655.20 (A down payment of \$500.00 plus 48 payments of \$357.40.)

II.

1.

a) Each existing square is divided into nine equal squares by a system of four line segments, just like a tic-tac-toe game. Of these nine newly formed squares, five are preserved (the four corner-squares and the central one) and four are discarded to yield the next iteration.

b) 625

c) $x_n = 5^n$ or
$f(n) = 5^n$ (Explicit)

$x_n = 5x_{n-1}$, with $x_0 = 1$, or
$f(n) = 5f(n-1)$, with $f(0) = 1$ (Recursive)

III.

1.

a) \$4359.88; \$6336.19

b) $A_n = (1 + .075/12)A_{n-1} + 100$, with $A_0 = 3000$. [Calculator: Enter 3000 and iterate $(1 + .075/12)ANS + 100$.]

c) A_{60} = \$11,612.59 (after 5 years)
A_{120} = \$24,129.29 (after 10 years)

d) 11 years and 10 months (You will have invested a total of \$17,200: the initial principal of \$3000, plus 142 payments of \$100.)

Notes